Economics 2010b: Microeconomic Theory II

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Time & Uncertainty

# 1 Time and Uncertainty in General Equilibrium

Another, and perhaps the most important, extension to the general equilibrium framework is to allow for both time and uncertainty. Introducing time into the framework is straightforward: we can think of a consumption good today as a different commodity than a consumption good tomorrow. Adding uncertainty turns out also to be straightforward due to an important modeling device of Arrow (1953): states of the world. A state of the world is a complete description of a date-event. Everyone agrees on the set of possible states and what state of the world is realized, although they need not necessarily agree on the probabilities of those states occurring. This way of thinking about uncertainty makes it very easy to extend the general equilibrium framework. In fact, Debreu's (1959) *Theory of Value* devotes only a short chapter to general equilibrium under uncertainty, and in some sense the first paragraph of that chapter tells us the main idea.

> "The analysis is extended in this chapter to the case where uncertain events determine the consumption sets, the production sets, and the resources of the economy. A contract for the transfer of a commodity now specifies, in addition to its physical properties, its location and its date, an event on the occurrence of which the transfer is conditional. This new definition of a commodity allows one to obtain a theory of uncertainty free from any probability concept and formally identical with the theory of certainty developed in the preceding chapters." (Debreu, 1959)

## 1.1 Arrow-Debreu Model

We will consider a parsimonious model of GE with uncertainty, although the framework can accommodate much more general specifications. Suppose there are $I$ consumers $i \in \mathcal{I}$, $L$ consumption goods $l \in \mathcal{L}$, and two periods, $t \in \{0, 1\}$. There are $S$ possible states of the world that can occur at $t = 1$, $s \in \mathcal{S} = \{1, \ldots, S\}$. A consumption bundle for consumer $i$ is an $x_i = (x_{0,i}, x_{1,i}, \ldots, x_{S,i})$, where $x_{0,i} = (x_{1,0,i}, \ldots, x_{L,0,i})$ is consumer $i$'s consumption of the $L$ goods at $t = 0$, and $x_{s,i} = (x_{1,s,i}, \ldots, x_{L,s,i})$ is her consumption of the $L$ goods at $t = 1$ and in state $s$. Her consumption set is $\mathcal{X}_i = \mathbb{R}_+^{L(S+1)}$, and her preferences are given by her utility function $U_i : \mathbb{R}_+^{L(S+1)} \to \mathbb{R}$. She has endowment $\omega_i = (\omega_{0,i}, \omega_{1,i}, \ldots, \omega_{S,i})$, where $\omega_{0,i} = (\omega_{1,0,i}, \ldots, \omega_{L,0,i})$ is her endowment of the $L$ goods at $t = 0$, and $\omega_{s,i} = (\omega_{1,s,i}, \ldots, \omega_{L,s,i})$ is her endowment of the $L$ goods at $t = 1$ in state $s$.

Since there is uncertainty, we also have to specify consumers' beliefs. Suppose, at $t = 0$, consumer $i$ believes that state $s \in \mathcal{S}$ will occur with probability $\pi_{s,i} \geq 0$, where $\sum_{s \in \mathcal{S}} \pi_{s,i} = 1$ for all $i$. Typically, we will think about consumers having the same beliefs, so that $\pi_{s,i} = \pi_s$ for all $i$, but the framework allows for subjective beliefs. We will also typically assume that consumers are expected utility maximizers with additively separable time preferences, so that we can write

$$U_i(x_i) = u_{0,i}(x_{0,i}) + \sum_{s \in \mathcal{S}} \pi_{s,i} u_{s,i}(x_{s,i}),$$

with $u_{0,i}$ and each of the $u_{s,i}$ functions concave.

In line with Debreu's description, we will think of each consumption good in each state of the world as being a separate commodity. To specify prices, therefore, we will have to specify $p = (p_0, p_1, \ldots, p_S)$, where $p_0 = (p_{1,0}, \ldots, p_{L,0}) \in \mathbb{R}^L$ is the price vector at $t = 0$, and $p_s = (p_{1,s}, \ldots, p_{L,s}) \in \mathbb{R}^L$ is the price vector at $t = 1$ in state $s \in \mathcal{S}$. That is, for a price $p_{l,s}$, consumers can buy and sell consumption of good $l$ in state $s$.

There are three important assumptions that allow us to make use of all of the results we have derived so far in this course. The first assumption is that all trade occurs at time

$t = 0$. So, at time $t = 0$, consumers buy and sell $t = 0$ commodities, and they also buy future claims to each commodity in each state of the world, and there is no opportunity for them to buy or sell at $t = 1$. The second important assumption is that the trading contracts that each consumer "writes" at $t = 0$ over $t = 1$ consumer are faithfully executed at $t = 1$. In the background, we are implicitly assuming the existence of an infallible third-party court system that perfectly compels consumers to execute their $t = 0$ contracts. This assumption, in turn, means that the third party can costlessly verify what state of the world was actually realized at $t = 1$. The third important assumption is that there is a market for each of the $L(S+1)$ state-contingent commodities.

Given prices $p$, consumer $i$ solves

$$\max_{x_i \in \mathbb{R}_+^{L(S+1)}} U_i(x_i) \text{ s.t. } x_i \in \mathcal{B}_i(p),$$

where consumer $i$'s budget set is given by

$$\mathcal{B}_i(p) = \left\{ x_i : p_0 \cdot x_{0,i} + \sum_{s \in \mathcal{S}} p_s \cdot x_{s,i} \leq p_0 \cdot \omega_{0,i} + \sum_{s \in \mathcal{S}} p_s \cdot \omega_{s,i} \right\}.$$

We will denote her Marshallian demand correspondence by $x_i(p, p \cdot x_i)$. A **pure-exchange economy with uncertainty** is therefore summarized by $\mathcal{E} = (u_i, \omega_i)_{i \in \mathcal{I}}$.

We are now in a position to define a Walrasian equilibrium in this context. For historical reasons, Walrasian equilibria in this model are referred to as Arrow-Debreu equilibria.

**Definition 10**. An **Arrow-Debreu equilibrium** for pure-exchange economy with uncertainty $\mathcal{E}$ is a vector $\left(p^*, (x_i^*)_{i \in \mathcal{I}}\right)$ that satisfies:

1. Consumer optimization: for all consumers $i \in \mathcal{I}$,

$$x_i^* \in \operatorname*{argmax}_{x_i \in \mathcal{B}_i(p^*)} U_i(x_i),$$

2. Market-clearing: for all commodities $l \in \mathcal{L}$ and all $s \in \{0, 1, \ldots, S\}$,

$$\sum_{i \in \mathcal{I}} x_{l,s,i}^{*} = \sum_{i \in \mathcal{I}} \omega_{l,s,i}.$$

This model is an elegant way of incorporating time and uncertainty into the basic framework because it allows us to apply all the results we have developed so far. For example, if $(A1) - (A4)$ hold, then a Walrasian equilibrium exists, and the welfare theorems hold.

The model does have some issues, though. One natural concern is that it seems unrealistic to think of all trading over future state-contingent commodities taking place at the beginning of time. Instead, we might expect that there would be different financial securities that are traded at potentially different times and these securities pay out when certain events occur. For example, car insurance pays out when your car is stolen, stocks pay dividends when a company is doing well, and so on.

**Exercise 15**. Consider a two date exchange economy with consumption at dates 0 and 1. There is a single consumer, one consumption good at each date, and there are $S$ states of the world (realized at date 1).

The consumer's utility function is

$$U = u(x_0) + \delta \sum_{s=1}^{S} \pi_s u(x_s),$$

where $x_0$ is date 0 consumption, $x_s$ is date 1 consumption in state $s$, $u$ is "well-behaved," and $\delta \in (0, 1)$. The consumer has initial endowment $(\omega_0, \omega_1, \ldots, \omega_S) \in \mathbb{R}_{+}^{S+1}$.

Write down the Arrow-Debreu equilibrium for this economy (normalize the price of date-0 consumption to be 1). Interpret the Arrow-Debreu relative prices: what factors determine whether they are high or low?

## 1.2 Sequential Trade and Arrow Equilibrium

Arrow later reformulated the model to allow for sequential trade in the following way. As before, there are two dates, $t \in \{0, 1\}$, and at date $t = 1$, a state of the world $s \in \mathcal{S}$ is realized. Suppose that consumption occurs only at $t = 1$, so that all consumers are endowed with $\omega_{l,0,i} = 0$ for all $l \in \mathcal{L}$, and $x_{0,i} \in \{0\}$ for all $i$. Moreover, suppose consumer $i$ is an expected utility maximizer, so that $U_i(x_i) = \sum_{s\in\mathcal{S}} \pi_{s,i} u_{s,i}(x_{s,i})$.

At date $t = 0$, consumers cannot directly trade all $L \cdot S$ state-contingent commodities. They can, however, trade securities that pay off different amounts in different states at $t = 1$. In particular, they can trade $S$ different Arrow securities, where at $t = 1$, **Arrow security** $s$ pays $1 if state $s$ is realized, and it pays 0 otherwise. Each consumer is endowed with 0 of each Arrow security, but they can have positive or negative holdings of them after trade occurs at $t = 0$. Denote by $z_i = (z_{1,i}, \ldots, z_{S,i})$ consumer $i$'s holdings of the $S$ Arrow securities, and denote the price vector for the Arrow securities by $q = (q_1, \ldots, q_S)$. To anticipate how we will think of more general securities in the next section, denote the **dividends vector for security** $k$ by $r_k = (r_{1,k}, \ldots, r_{S,k}) \in \mathbb{R}_+^S$, where $r_{s,k}$ is the amount that security $k$ pays in state $s$. The dividends vector for Arrow security 1 is therefore $r_1^A \equiv (1, 0, \ldots, 0)$, and for Arrow security $k$ is $r_k^A \equiv (0, \ldots, 0, 1, 0, \ldots, 0)$, where the $k$th element is 1 and all others are 0.

At $t = 1$, the state of the world $s$ is realized, and then markets for each of the $L$ goods open, consumers trade at prices $p_s = (p_{s,1}, \ldots, p_{s,L})$, and then they consume.

Under this specification, given Arrow security prices $q$ and goods prices $p$, consumer $i$ solves the following problem:

$$\max_{(z_i,x_i)} \sum_{s\in\mathcal{S}} \pi_{s,i} u_{s,i}(x_{s,i}) \text{ s.t. } (z_i, x_i) \in \mathcal{B}_i(q, p),$$

where her budget set is now given by

$$\mathcal{B}_i(q, p) = \{(z_i, x_i) : q \cdot z_i \leq 0, p_s \cdot x_{s,i} \leq p_s \cdot \omega_{s,i} + z_{s,i} \text{ for all } s \in \mathcal{S}\}.$$

The first inequality in the definition of the budget set reflects the assumption that the consumer is not endowed with any Arrow securities, so that the net value of the Arrow securities she holds after $t = 0$ trade has to be nonpositive. The second set of inequalities reflects her budget set at $t = 1$ in state $s$. Her wealth in state $s$ is the sum of the wealth from her endowment, $p_s \cdot \omega_{s,i}$, and the wealth she obtains from her Arrow securities, $z_{s,i}$, which can be positive or negative. Note that, since we are assuming that $x_i \in \mathbb{R}_+^{L \cdot S}$, we are implicitly imposing the constraint that each consumer has nonnegative wealth at $t = 1$: $z_{s,i} \geq -p_s \cdot \omega_{s,i}$.

The idea behind this alternative setup of the model is that consumers will trade multiple times, and their wealth each time they trade is determined by their endowment in the "spot market" as well as how much they loaned and borrowed. Consumers correctly anticipate spot-market prices in each state at $t = 0$, even though they cannot trade in those markets until $t = 1$, and they buy and sell Arrow securities to transfer their wealth from one state to the next so they can buy the commodities they would like to buy in those states. We will refer to the economy as a **sequential-exchange economy with a complete set of Arrow securities** and denote it by $\mathcal{E}^{SE} = \left( (u_i, \omega_i)_{i \in \mathcal{I}}, \left( r_k^A \right)_{k \in \mathcal{S}} \right)$, where $r_k^A$ is the returns vector for the $k$th Arrow security.

We can now define our notion of Walrasian equilibrium in this setting.

**Definition 11**. An **Arrow equilibrium** for a sequential-exchange economy with a complete set of Arrow securities, $\mathcal{E}^{SE}$, is a vector $\left( q^*, p^*, (z_i^*, x_i^*)_{i \in \mathcal{I}} \right)$ of Arrow security prices and state-contingent consumption-good prices and Arrow security positions and consumption bundles for each consumer $i \in \mathcal{I}$ that satisfies:

1. Consumer optimization: for all consumers $i \in \mathcal{I}$,

$$(z_i^*, x_i^*) \in \underset{(z_i, x_i) \in \mathcal{B}_i(q,p)}{\operatorname{argmax}} \sum_{s \in \mathcal{S}} \pi_{s,i} u_{s,i} \left( x_{s,i} \right),$$

2. Market-clearing: $\sum_{i \in \mathcal{I}} z_i^* = 0$ and, for all commodities $l \in \mathcal{L}$ and all $s \in \mathcal{S}$,

$$\sum_{i \in \mathcal{I}} x_{l,s,i}^* = \sum_{i \in \mathcal{I}} \omega_{l,s,i}.$$

Given this definition of equilibrium, we can now describe the main result of this section, which links the set of allocations that can arise in an Arrow-Debreu equilibrium in a pure-exchange economy with uncertainty, $\mathcal{E} = (u_i, \omega_i)_{i\in\mathcal{I}}$ to the set of allocations that can arise in an Arrow equilibrium in a sequential-exchange economy with a complete set of Arrow securities, $\mathcal{E}^{SE} = \left((u_i, \omega_i)_{i\in\mathcal{I}}, \left(r_k^A\right)_{k\in\mathcal{S}}\right)$.

**Theorem 13 (Equivalence of Arrow and Arrow-Debreu equilibrium)**. Given economies $\mathcal{E}$ and $\mathcal{E}^{SE}$ with the same consumer preferences and endowments, $(x_i^*)_{i\in\mathcal{I}}$ is an Arrow-Debreu equilibrium allocation for $\mathcal{E}$ if and only if, for some $(z_i^*)_{i\in\mathcal{I}}$, $(z_i^*, x_i^*)_{i\in\mathcal{I}}$ is an Arrow equilibrium allocation for $\mathcal{E}^{SE}$.

**Proof of Theorem 13**. Take an Arrow equilibrium $\left(q^*, p^*, (z_i^*, x_i^*)_{i\in\mathcal{I}}\right)$ for economy $\mathcal{E}^{SE}$. By monotonicity of preferences, we will have that for each consumer $i \in \mathcal{I}$, $q^* \cdot z_i^* (q^*, p^*) = 0$ and $z_{s,i}^* (q^*, p^*) = p_s^* \cdot x_{s,i}^* (q^*, p^*) - p_s^* \cdot \omega_{s,i}$. We can combine these two equations to get

$$\sum_{s\in\mathcal{S}} q_s^* \left(p_s^* \cdot x_{s,i}^* - p_s^* \cdot \omega_{s,i}\right) = 0.$$

The consumption bundle $x_{s,i}^*$ therefore solves the problem:

$$\max_{x_i} \sum_{s\in\mathcal{S}} \pi_{s,i} u_{s,i} (x_{s,i})$$

subject to

$$\sum_{s\in\mathcal{S}} (q_s^* p_s^*) \cdot x_{s,i} \leq \sum_{s\in\mathcal{S}} (q_s^* p_s^*) \cdot \omega_{s,i}.$$

Define state-dependent prices $\hat{p}_s^* \in \mathbb{R}^L$ for $s \in \mathcal{S}$ with $\hat{p}_s^* \equiv q_s^* p_s^*$. Then, $\left(\hat{p}^*, (x_i^*)_{i\in\mathcal{I}}\right)$ is an Arrow-Debreu equilibrium for economy $\mathcal{E}$.

Going the other direction, suppose $\left(p^*, (x_i^*)_{i\in\mathcal{I}}\right)$ is an Arrow-Debreu equilibrium for $\mathcal{E}$. Then $\left(q^*, \hat{p}^*, (z_i^*, x_i^*)_{i\in\mathcal{I}}\right)$, where $q_s^* = 1$, $\hat{p}^* = p^*$, and

$$z_{s,i}^* = p_s^* \cdot x_{s,i}^* - p_s^* \cdot \omega_{s,i}$$

is an Arrow equilibrium for economy $\mathcal{E}^{SE}$.■

This theorem establishes an equivalence between the notion of Arrow-Debreu equilibrium in which trade in all $L \cdot S$ markets occurs ex ante and the notion of Arrow equilibrium, in which trade occurs in only $S$ markets at $t = 0$ and in $L$ markets at $t = 1$. One disadvantage of the notion of Arrow equilibrium is that even though trading seems less complicated, in a sense, consumers still must form consistent expectations about what the equilibrium goods prices will be at $t = 1$ when they are trading securities at $t = 0$. That said, one of the big advantages of the sequential exchange framework is that it allows us to investigate what happens when the $t = 0$ securities market does not have a complete set of Arrow securities. That is, what happens if markets are *incomplete*? We will turn to this question now.

**Exercise 16**. There are two farmers, named Octavia and Seema, who can trade only with each other. In years when there is no flood, both farms yield 10 units of corn; in years when there is a flood, Octavia's farm yields 10 units of corn, and Seema's farm yields 5 units of corn. The probability of a flood is given by $\pi = 1/2$, which is common knowledge to the farmers. The farmers have identical utility functions given by $u(x) = \ln(x)$, where $x$ is the units of corn consumed.

($a$) Suppose that Octavia and Seema set up an exchange market to securitize corn at the beginning of the year (before knowing the realization of the state of the world). Compute the equilibrium prices and allocations.

Suppose that Seema has the option of building a greenhouse at a cost before realizing the state of the world. If she builds a greenhouse, Seema's farm will produce 10 units of corn in all states of the world.

($b$) Using the equilibrium results computed above, how much would each farmer be willing to pay for the greenhouse? Assume that each considers paying for the greenhouse entirely by herself. In this context, should we consider the possibility of "negative" willingness-to-pay? That is, might one farmer be willing to pay the other not to build the greenhouse?

($c$) Would your above answer change if Octavia and Seema were unable to trade ex post (after the state of the world is realized)? If so, how? Would your answer change if they were unable to trade ex ante (there is no exchange market to securitize corn at the beginning of the year)?

## 1.3 Incomplete Markets

When we talked about sequential exchange economies in the previous section, we assumed that there was a complete set of Arrow securities that could be traded at $t = 0$. One important implication of this assumption that we did not emphasize is that it allowed consumers to insure themselves against the state of the world by transferring wealth from states in which their marginal utility of income is low (either because they do not especially value consumption in such states or because their endowment in such states is high) to states in which their marginal utility of income is high. In an Arrow equilibrium, the resulting risk-sharing is efficient, since the first welfare theorem applies in that setting.[1] In contrast, when markets are *incomplete*, risk sharing in the economy will generally be inefficient. This will imply that Walrasian equilibrium allocations in such economies are not Pareto optimal.

Suppose there are $K$ **securities** $k \in \mathcal{K} = \{1, \ldots, K\}$, where security $k$ has **dividends vector** $r_k = (r_{1,k}, \ldots, r_{S,k}) \in \mathbb{R}^S_+$. We can think of each security as a share in a company that pays out dividends $r_{s,k}$ in state $s$. If consumer $i$ owns **portfolio** $z_i = (z_{1,i}, \ldots, z_{K,i})$, then in state $s$, her wealth will be $p_s \cdot \omega_{s,i} + \sum_{k \in \mathcal{K}} z_{k,i} r_{s,k}$. If we denote the **dividends matrix** $R = \left(r_1^T, \ldots, r_K^T\right)$, where $r_k^T$ is the transpose of $r_k$, then we will say that the securities market is **incomplete** if $rank\,(R) < S$. Otherwise, we will say that the securities market is **complete**. If there is a complete set of Arrow securities, then $R$ is the $S \times S$ identity matrix, and the securities market is complete.

As in the previous subsection, given security prices $q = (q_1, \ldots, q_K)$ and goods prices $p$, consumer $i \in \mathcal{I}$ solves the following problem:

$$\max_{(z_i, x_i)} \sum_{s \in \mathcal{S}} \pi_{s,i} u_{s,i}\left(x_{s,i}\right) \text{ s.t. } \left(z_i, x_i\right) \in \mathcal{B}_i\left(q, p\right),$$

[1] More precisely, define $v_{s,i}\left(z_{s,i}\right) = \max_{x_{s,i}} u_{s,i}\left(x_{s,i}\right)$ subject to $p_s \cdot x_{s,i} \leq p_s \cdot \omega_{s,i} + z_{s,i}$ to be consumer $i$'s indirect utility in state $s$ when she has $z_{s,i}$ units of Arrow security $s$. If we assume $v_{s,i}$ is concave and differentiable, then Pareto optimality ensures that for all $i, i'$, $\frac{\pi_{s,i} \partial v_{s,i} / \partial z_{s,i}}{\pi_{s,i'} \partial v_{s,i'} / \partial z_{s,i'}} = \frac{\lambda_i}{\lambda_{i'}}$ for all $s$. That is, the ratio of marginal utilities of income are equalized across states for any two consumers. By the first welfare theorem, any Arrow equilibrium allocation satisfies these properties.

where her budget set is now given by

$$\mathcal{B}_i(q,p) = \left\{ (z_i, x_i) : q \cdot z_i \leq 0, p_s \cdot x_{s,i} \leq p_s \cdot \omega_{s,i} + \sum_{k \in \mathcal{K}} z_{k,i} r_{s,k} \text{ for all } s \in \mathcal{S} \right\}.$$

As before, consumers maximize their expected utility subject to a $t = 0$ budget constraint and a $t = 1$ budget constraint for each state $s \in \mathcal{S}$. The first inequality in the definition of the budget set again reflects the assumption that the consumer is not endowed with any securities.

A **sequential-exchange economy with securities** $\mathcal{K}$ is summarized by a vector $\mathcal{E}^{SE} = \left((u_i, \omega_i)_{i \in \mathcal{I}}, R\right)$. We can now define our notion of Walrasian equilibrium for such an economy.

**Definition 12**. An **incomplete-markets equilibrium (or Radner equilibrium)** for a sequential-exchange economy with securities $\mathcal{K}$ is a vector $\left(q^*, p^*, (z_i^*, x_i^*)_{i \in \mathcal{I}}\right)$ of security prices and state-contingent consumption-good prices and security positions and consumption bundles for each consumer $i \in \mathcal{I}$ that satisfies:

1. Consumer optimization: for all consumers $i \in \mathcal{I}$,

$$(z_i^*, x_i^*) \in \underset{(z_i, x_i) \in \mathcal{B}_i(q,p)}{\operatorname{argmax}} \sum_{s \in \mathcal{S}} \pi_{s,i} u_{s,i}(x_{s,i}),$$

2. Market-clearing: $\sum_{i \in \mathcal{I}} z_{k,i}^* = 0$ for all $k \in \mathcal{K}$, and, for all commodities $l \in \mathcal{L}$ and all $s \in \mathcal{S}$,

$$\sum_{i \in \mathcal{I}} x_{l,s,i}^* = \sum_{i \in \mathcal{I}} \omega_{l,s,i}.$$

In general, when markets are incomplete, a Radner equilibrium need not exist, and even if it does exist, the resulting allocation will typically not be Pareto optimal. If, however, $L = 1$, so there is only a single consumption good, then a Radner equilibrium exists, and it does have some optimality properties.

For $L = 1$, Diamond (1967) showed that a Radner equilibrium exists by showing that the consumer optimization problem boils down to a more familiar problem. In particular, at any solution to consumer $i$'s problem, we must have $q \cdot z_i = 0$ and $x_{s,i} = \omega_{s,i} + \sum_{k \in \mathcal{K}} z_{k,i} r_{s,k}$. We can substitute this second constraint into the consumer's problem, which then becomes

$$\max_{z_i} \underbrace{\sum_{s \in \mathcal{S}} \pi_{s,i} u_{s,i} \left( \omega_{s,i} + \sum_{k \in \mathcal{K}} z_{k,i} r_{s,k} \right)}_{\equiv \tilde{u}_i(z_i)}$$

subject to $q \cdot z_i \leq 0$. Diamond pointed out that such an economy is equivalent, in some sense, to an economy in which consumer preferences are given by $\tilde{u}_i (z_i)$ and there are $K$ "commodities"—one corresponding to each of the securities. So as long as $\tilde{u}_i$ satisfies $(A1)$ – $(A3)$, then a WE exists. The interior endowments assumption $(A4)$ is not necessary for the existence result because consumers are allowed to "consume" negative quantities of $z_i$.

Such equilibria need not yield Pareto-optimal allocations. To see why, consider an example in which $L = 1, K = 1$, and $S = 2$. The security pays 1 in each state of the world. There are two consumers with endowments $\omega_1 = (2, 1)$ and $\omega_2 = (1, 2)$, so that consumer 1 is endowed with one unit of the consumption good in state 1 and 2 units in state 2. Both consumers have identical preferences given by

$$u_i (x_{1,i}, x_{2,i}) = \frac{1}{2} \log x_{1,i} + \frac{1}{2} \log x_{2,i}.$$

As an exercise, it is worth verifying that there is a unique Radner equilibrium of this economy. In this equilibrium, there will be no trade in the security at $t = 0$, and consumers will consume their endowments at $t = 1$. This allocation is Pareto dominated by the feasible allocation $x_1 = x_2 = (3/2, 3/2)$.

The first welfare theorem fails in this situation because the set of existing securities does not allow the consumers to insure themselves against states in which they will have a low endowment. Nevertheless, there is still a sense in which the resulting allocation exhausts the gains from trade and is therefore what we refer to as constrained efficient.

**Definition 13.** Given endowments $(\omega_i)_{i\in\mathcal{I}}$ and securities $\mathcal{K}$, an allocation $(x_i)_{i\in\mathcal{I}}$ is **constrained efficient** if $\sum_{i\in\mathcal{I}} x_i \leq \sum_{i\in\mathcal{I}} \omega_i$, and for all $i$, there exists $z_i \in \mathbb{R}^K$ such that $x_i = \omega_i + Rz_i$, and there exists no alternative allocation $(\hat{x}_i)_{i\in\mathcal{I}}$ that Pareto dominates $(x_i)_{i\in\mathcal{I}}$ and also satisfies $\sum_{i\in\mathcal{I}} \hat{x}_i \leq \sum_{i\in\mathcal{I}} \omega_i$ and $\hat{x}_i = \omega_i + Rz_i$ for some $z_i \in \mathbb{R}^K$ for all $i$.

When there is only a single consumption good and consumers have monotone preferences, Radner equilibrium allocations are always constrained efficient, as the following theorem illustrates.

**Theorem 14.** If $\mathcal{E}^{SE}$ has $L = 1$ and satisfies $(A2)$, then if $\left(q^*, p^*, (z_i^*, x_i^*)_{i\in\mathcal{I}}\right)$ is a Radner equilibrium, $(x_i^*)_{i\in\mathcal{I}}$ is constrained efficient.

We will conclude this section with a few comments on the generality of this theorem. As Hart (1975) illustrates, when $L = 2$, there may exist Radner equilibria that are not constrained efficient (see, for instance, MWG Example 19.F.2). Also, when markets are incomplete, weird things can happen. For example, adding another security that is linearly independent of existing securities can actually make all consumers strictly worse off (see, for instance, MWG Exercise 19.F.3). Finally, when markets are incomplete, Geanakoplos and Polemarchakis (1986) shows that it is generically true that a social planner can improve efficiency by introducing a small tax or subsidy. This is an illustration of the "general theory of the second-best" (Lipsey and Lancaster, 1963): when there is an unresolvable market failure (market incompleteness, in this case), it is generically the case that there exists a further distortion that a social planner could conceivably put in place that leads to a more efficient allocation.