Economics 2010b: Microeconomic Theory II

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Existence; Uniqueness; Stability; Testability

1 Existence of Walrasian Equilibrium

Last week, we focused on the normative, efficiency properties of Walrasian equilibria. This week, we will focus on a couple positive properties. In particular, we will begin by asking what seems like a straightforward question: does a Walrasian equilibrium exist? And then we will ask a few other important questions relating to equilibrium uniqueness, equilibrium stability, and the comparative statics of Walrasian equilibria.

The question of whether a Walrasian equilibrium exists really boils down to: under what conditions on preferences and endowments does a Walrasian equilibrium exist? We know from the example in Figure 5(b) from last week that a Walrasian equilibrium does not always exist. And we know from the second welfare theorem that when assumptions $(A1)-(A4)$ are satisfied, then if the endowment is a Pareto-optimal allocation, there is a Walrasian equilibrium for which it is the equilibrium allocation. In some sense, the second welfare theorem provides a bit of a mundane answer to the existence question, since it provides conditions under which no trade is optimal for each consumer. The more interesting question is the more difficult one: when is a Walrasian equilibrium guaranteed to exist if the endowment itself is not already Pareto optimal? That is, when is there a Walrasian equilibrium that actually involves trade?

This was an open question ever since Walras's formulation of the GE model in the 1870's until Arrow, Debreu, and McKenzie produced the first rigorous existence proofs in the 1950's. The basic question is, given aggregate demand functions $\sum_{i\in\mathcal{I}} x_{l,i}\left(p, p\cdot\omega_i\right)$ for each commodity, when does there exist a price vector p^* such that $\sum_{i\in\mathcal{I}} x_{l,i}\left(p^*, p^*\cdot\omega_i\right) = \sum_{i\in\mathcal{I}}\omega_{l,i}$ for all $l\in\mathcal{L}$? Early arguments amounted to just counting up the number of equations and unknowns, but these approaches were not satisfactory, since it would not be clear what would happen if the solution to the equations involved negative prices or quantities. The breakthrough came in the 1950's when Arrow and Debreu (1954) proved the following existence result.

Theorem 4 (Existence of Walrasian Equilibrium). Given an economy $\mathcal{E}$ satisfying $(A1)-(A4)$, there exists a Walrasian equilibrium $\left(p^*, (x_i^*)_{i\in\mathcal{I}}\right)$.

The key insight in the 1950's was to reframe the Walrasian equilibrium existence question as a *fixed-point* question, following John Nash's (1951) proof of the existence of Nash equilibrium using a related approach. A *fixed point* of a correspondence $f : Z \rightrightarrows Z$ is a point z such that $z \in f(z)$, and fixed-point theorems provide fairly general conditions under which functions or correspondences have fixed points.

The important step in making use of the general-purpose technology of fixed-point theorems is to figure out how to map the equilibrium existence question into the question of whether a suitably chosen correspondence has a fixed point: it is about choosing the right correspondence. Suppose the correspondence f maps an allocation $(x_i)_{i\in\mathcal{I}}$ and a price vector p into a new allocation $(x_i')_{i\in\mathcal{I}}$ and price vector p', where the new allocation is the set of optimal choices for consumers given the price vector p, and the new price vector p' is one that raises the prices of over-demanded goods and lowers the price of under-demanded goods under the allocation $(x_i)_{i\in\mathcal{I}}$ and otherwise does not change prices. Then a fixed point of f will be a Walrasian equilibrium, so if we can show that f satisfies the conditions required for a fixed-point theorem to apply, then we can conclude that a Walrasian equilibrium exists.

1.1 Two-Commodity Intuitive Sketch

We will first go through an intuitive argument for equilibrium existence in the special case of two commodities, and then we will go through the more general result described in Theorem 4. The argument in the two-commodity case will also develop some tools that will be useful when we talk about uniqueness and stability of Walrasian equilibrium. For this part, we will strengthen the monotonicity condition $(A2)$ to a strong monotonicity condition $(A2')$.

Assumption A2' (strong monotonicity). For all consumers $i \in \mathcal{I}$, u_i is strictly increasing: $u_i(x_i') > u_i(x_i)$ whenever $x_{l,i}' \geq x_{l,i}$ for all $l \in \mathcal{L}$ with at least one inequality strict.

As a starting point, we are going to introduce the idea of an excess demand function for an economy $\mathcal{E} = (u_i, \omega_i)_{i \in \mathcal{I}}$. The **excess demand function for consumer** i is $z_i(p) = x_i(p, p \cdot \omega_i) - \omega_i$. The **aggregate excess demand function** is the sum of consumers' excess demand functions $z(p) = \sum_{i \in \mathcal{I}} z_i(p)$. It should be clear from the definition of the aggregate excess demand function that if there is a p^* that satisfies $z(p^*) = 0$, then $\left(p^*, (x_i^*)_{i \in \mathcal{I}}\right)$ with $x_i^* = x_i(p^*, p^* \cdot \omega_i)$ is a Walrasian equilibrium. The way x_i^* is defined, it is clear that at p^*, x_i^* is consumer i's optimal consumption bundle. Moreover, if $z(p^*) = 0$, then markets clear for each commodity. In this case, proving existence boils down to establishing that a solution to $z(p) = 0$ exists given assumptions $(A1), (A2'), (A3)$, and $(A4)$. The aggregate excess demand function inherits many of the properties of Marshallian demand functions, as the next lemma illustrates.

Lemma 3. Suppose $\mathcal{E}$ satisfies $(A1), (A2'), (A3)$,$(A4)$, and $\mathcal{X}_i = \mathbb{R}_+^L$ for all i. Then the aggregate excess demand function satisfies:

(i) z is continuous;

(ii) z is homogeneous of degree zero;

(iii) $p \cdot z(p) = 0$ for all p (Walras's Law);

(iv) there is some $Z > 0$ such that $z_l(p) > -Z$ for every $l \in \mathcal{L}$ and for every p;

(v) if $p^n \to p$, where $p \neq 0$ and $p_l = 0$ for some l, then $\max\{z_1(p^n), \ldots, z_L(p^n)\} \to \infty$.

The first property is something that was assumed in Economics 2010a, but it is straightforward to show that it follows from assumptions $(A1) - (A4)$.[1] The second property is

[1] z is upper hemi-continuous from Berge's maximum theorem, it is non-empty because preferences are continuous, and it is convex-valued because preferences are convex. These properties imply that z is a continuous correspondence.

straightforward, and we already proved the third property in week 1. The fourth property follows directly from the assumption that $\mathcal{X}_i = \mathbb{R}^L_+$.

The last property bears some comment. It is saying that as some, but not all, prices go to zero, there must be some consumer whose wealth is not going to zero. Because she has strongly monotone preferences, she must demand more of one of the commodities whose price is going to zero.

To gain intuition for the general existence proof, let us consider the case where there are only two goods in the economy, and let us further assume that consumer preferences are strictly concave, so that $x_i(p, p \cdot \omega_i)$ is a singleton for all p (we will allow for $x_i(p, p \cdot \omega_i)$ to be a correspondence—for there to be multiple optimal allocations for a given consumer at a given price vector—when we prove the general theorem). Our goal is to find a price vector $p = (p_1, p_2)$ for which $z(p) = 0$. Because $z(\cdot)$ is homogeneous of degree zero, we can normalize one of the prices, say p_2, to one. This reduces our search to price vectors of the form $(p_1, 1)$. Moreover, Walras's Law implies that if the market for commodity 1 clears, then so does the market for commodity 2, so it suffices to find a price p_1 such that $z_1(p_1, 1) = 0$.

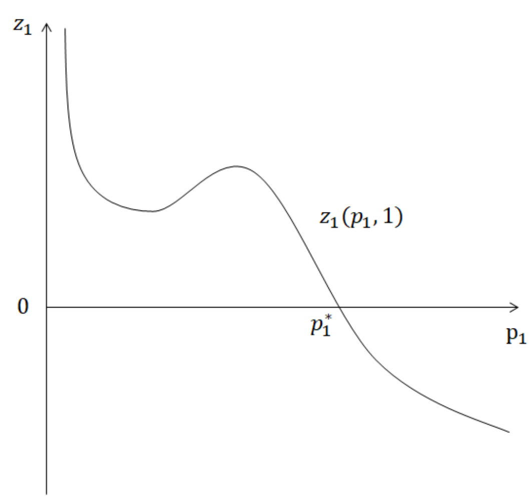

Figure 10: Existence of WE with two commodities

The problem of finding a p_1 such that $z_1(p_1, 1) = 0$ is a one-dimensional problem, so we can just graph it. Figure 10 plots $z_1(p_1, 1)$ as a function of p_1. The figure highlights three important properties of $z_1(p_1, 1)$. First, it is continuous. Second, for p_1 very small, $z_1(p_1, 1) > 0$, and third, for p_1 very large, $z_1(p_1, 1) < 0$. Given these three properties, by the intermediate value theorem—the simplest of fixed-point theorems—there necessarily exists some $p^* = (p_1^*, 1)$ such that $z_1(p^*) = 0$, and a Walrasian equilibrium therefore exists. The subtleties in making this argument are in establishing that $z_1(p_1, 1) > 0$ for p_1 small and $z_1(p_1, 1) < 0$ for p_1 large. The first property follows from condition (v) in the Lemma above. The second property follows because if $p_1 \to \infty$, then each consumer's demand for commodity 1 will converge to something less than her endowment of commodity 1, as continuity and monotonicity of preferences imply she would like to sell at least some of commodity 1 for an unboundedly large amount of commodity 2.

Exercise 10 (Adapted from MWG 17.C.4). Consider a pure exchange economy. The only novelty is that a progressive tax system is instituted according to the following rule: individual wealth is no longer $p \cdot \omega_i$; instead, anyone with wealth above the mean of the population must contribute half of the excess over the mean into a fund, and those below the mean receive a contribution from the fund in proportion to their deficiency below the mean.

(a) For a two-consumer society with endowments $\omega_1 = (1, 2)$ and $\omega_2 = (2, 1)$, write the after-tax wealths of the two consumers as a function of prices.

(b) If the consumer preferences are continuous, strictly convex, and strongly monotone, will the excess demand functions satisfy the conditions required for existence stated in Lemma 3?

1.2 More General Existence Result

Before proving the main existence theorem, we will first remind ourselves of a couple important mathematical theorems that we will be using in the proof. The first is the Kakutani fixed-point theorem, which you used to prove the existence of a Nash equilibrium in Economics 2010a. The second is the maximum theorem or Berge's maximum theorem.

Kakutani Fixed-Point Theorem. Suppose $\mathcal{Z}$ is a nonempty, compact, convex subset of $\mathbb{R}^n$ and that $f : \mathcal{Z} \rightrightarrows \mathcal{Z}$ is a nonempty, convex-valued, and upper hemi-continuous correspondence. Then f has a fixed point.

Kakutani's fixed-point theorem is a generalization of Brouwer's fixed-point theorem but for set-valued functions. The basic idea of the theorem is that a fixed point is an intersection of the graph of f with the 45° line, and the conditions for the theorem ensure that the graph of f cannot "jump" across the 45° line. In the special case when $n = 1$ and when f is scalar-valued, this theorem boils down to the intermediate-value theorem.

The proof of equilibrium existence is going to make use of the Kakutani fixed-point theorem for an appropriately defined correspondence, and we will need to be able to establish that the correspondence has the properties that are required by the theorem. The following theorem will be useful for establishing these properties.

Berge's Maximum Theorem. If $f : \mathcal{X} \times \Theta \to \mathbb{R}$ is a continuous function, and $C : \Theta \rightrightarrows \mathcal{X}$ is a continuous, compact-valued correspondence, then $V(\theta) = \max\{f(x, \theta) : x \in C(\theta)\}$ is continuous in θ, and $X^*(\theta) = \operatorname{argmax}\{f(x, \theta) : x \in C(\theta)\}$ is non-empty, compact-valued, and upper hemi-continuous.

The line of proof we will be following is to define a set $\mathcal{Z}$ and a correspondence $f : \mathcal{Z} \rightrightarrows \mathcal{Z}$ that satisfies the conditions of Kakutani fixed-point theorem and whose fixed points are Walrasian equilibria. There are therefore three main questions we will need to answer:

1. What should $\mathcal{Z}$ and f be?
2. Why do the conditions of Kakutani's fixed-point theorem hold?
3. Why do the fixed points of f correspond to Walrasian equilibria?

1.2.1 Step 1: Define $\mathcal{Z}$ and f

To define the set $\mathcal{Z}$, it is convenient to first normalize prices so that they sum to one. Define the normalized price simplex Δ to be the set of associated price vectors: $\Delta \equiv$

$\left\{p \in \mathbb{R}^L_+ : \sum_{l \in \mathcal{L}} p_l = 1\right\}$. Next, for each consumer i, define a non-empty, compact, convex subset of her consumption set that is bounded above by what she could consume if she possessed the entire aggregate endowment: $\mathcal{T}_i = \left\{x_i \in \mathcal{X}_i : x_i \leq 2\sum_{i \in \mathcal{I}} \omega_i\right\} \subset \mathcal{X}_i$. Since each $\mathcal{T}_i$ is a compact, so is the product set $\mathcal{T} = \prod_{i \in \mathcal{I}} \mathcal{T}_i$. Define $\mathcal{Z} \equiv \mathcal{T} \times \Delta$ to be the domain on which we will define the correspondence f.

The correspondence $f : \mathcal{Z} \rightrightarrows \mathcal{Z}$ will map an allocation $(x_i)_{i \in \mathcal{I}}$ and a price vector p to the set of allocations $(x_i')_{i \in \mathcal{I}}$ that are optimal for each consumer given p and a new price vector p' that raises the price of commodities that were over-demanded and lowers the price of commodities that were under-demanded under $(x_i)_{i \in \mathcal{I}}$. The first part of this construction is straightforward. Let $x_i^{\mathcal{T}}(p, p \cdot \omega_i)$ be consumer i's optimal choice over $\mathcal{B}_i(p) \cap \mathcal{T}_i$. Marshallian demand correspondence at prices p:

$$x_i^{\mathcal{T}}(p, p \cdot \omega_i) = \max_{x_i \in \mathcal{B}_i(p) \cap \mathcal{T}_i} u_i(x_i),$$

where recall that $\mathcal{B}_i(p)$ is consumer i's budget set given prices p:

$$\mathcal{B}_i(p) = \{x_i \in \mathcal{X}_i : p \cdot x_i \leq p \cdot \omega_i\}.$$

If we do this for each consumer, we get the product of correspondences $\prod_{i \in \mathcal{I}} x_i^{\mathcal{T}}(p, p \cdot \omega_i) \subset \mathcal{T}$. This takes care of the first part of the construction.

For the second part of the construction, we introduce a fictitious "player" called the Walrasian auctioneer (or the "price player") who chooses a price vector $p \in \mathbb{R}^L_+$ and wants to maximize the value of aggregate excess demand. Let

$$a^*(x) = \operatorname*{argmax}_{\tilde{p} \in \Delta} \tilde{p} \cdot \left(\sum_{i \in \mathcal{I}} x_i - \sum_{i \in \mathcal{I}} \omega_i\right),$$

where "a^*" is a pneumonic for "auctioneer." We are now in a position to define the appropriate correspondence $f : \mathcal{Z} \rightrightarrows \mathcal{Z}$ by

$$f(x,p) = \underbrace{\left(\prod_{i \in \mathcal{I}} x_i^{\mathcal{T}}(p, p \cdot \omega_i)\right)}_{\subset \mathcal{T}} \times \underbrace{a^*(x)}_{\subset \Delta}.$$

1.2.2 Step 2: Verify that Kakutani's theorem can be applied

Now that we have defined the set $\mathcal{Z}$ and the correspondence f, we will verify that the conditions of Kakutani's fixed-point theorem hold, so it can be applied. The first set of conditions that needs to be verified is that $\mathcal{Z} = \mathcal{T} \times \Delta$ is a non-empty, compact, convex subset of $\mathbb{R}^n$ for some n. The second set of conditions is on the correspondence f—we need to show that f is a non-empty, convex-valued, and upper hemi-continuous correspondence. Note that the product of non-empty, convex-valued, and upper hemi-continuous correspondences is itself non-empty, convex-valued, and upper hemi-continuous, so this last part requires that we show that each of the correspondences $x_i^{\mathcal{T}}(p, p \cdot \omega_i)$ and $a^*(x)$ satisfy these conditions.

First, note that $\mathcal{Z}$ is a non-empty, compact, and convex subset of $\mathbb{R}^{I \cdot L + L}$ because each $\mathcal{T}_i$ and Δ are non-empty, compact, and convex subsets of $\mathbb{R}^L$.

Next, a^* is non-empty and convex-valued because Δ is non-empty, compact, and convex, and the Walrasian auctioneer's objective is linear in p and hence continuous. It is upper hemi-continuous by Berge's maximum theorem.

Finally, the function $x_i^{\mathcal{T}}$ is non-empty and convex-valued because $\mathcal{B}_i(p) \cap \mathcal{T}_i$ is non-empty, compact, and convex, and u_i is continuous (guaranteeing x_i^* is non-empty) and concave (guaranteeing x_i^* is convex-valued). These conditions alone are not enough to give us the upper hemi-continuity that we require in order to apply Kakutani's fixed-point theorem, however, because we still have to show that $\mathcal{B}_i(p) \cap \mathcal{T}_i$ is a continuous, compact-valued correspondence.

It is apparent that $\mathcal{B}_i(p) \cap \mathcal{T}_i$ is compact-valued—the involved part is showing that it is a continuous correspondence. To do so, we have to show that it is both upper hemi-continuous in p and lower hemi-continuous in p. Upper hemi-continuity is straightforward, since if $p^n \to p$ and $x_i^n \to x_i$ with $x_i^n \in \mathcal{B}_i(p^n) \cap \mathcal{T}_i$ for all n, then $p^n \cdot x_i^n \leq p^n \cdot \omega_i$ and $x_i^n \leq \sum_{i \in \mathcal{I}} \omega_i$ for all n and therefore this condition holds in the limit as well. Showing that $\mathcal{B}_i(p) \cap \mathcal{T}_i$ is lower hemi-continuous is more involved, and we leave this as an exercise (this is the only part of the existence proof that makes use of assumption $(A4)$).

These arguments establish that $\mathcal{Z}$ and f satisfy the conditions of Kakutani's fixed-point theorem, and therefore f has a fixed point. We now need to show that any such fixed point of f is a Walrasian equilibrium.

1.2.3 Step 3: Show that fixed points of f are Walrasian equilibria

Suppose $(x^*, p^*) \in f(x^*, p^*)$—that is, (x^*, p^*) is a fixed point of f. We need to show that $x_i^* = x_i^{\mathcal{T}}(p^*, p^* \cdot \omega_i)$ is consumer-optimal for each $i \in \mathcal{I}$, and markets clear at prices p^*.

For the first part, because $x_i^{\mathcal{T}}(p^*, p^* \cdot \omega_i) = \max_{x_i \in \mathcal{B}_i(p^*) \cap \mathcal{T}_i} u_i(x_i)$, we need to verify that the resulting solution also solves the relaxed problem $\max_{x_i \in \mathcal{B}_i(p^*)} u_i(x_i)$, which is the problem the consumer actually faces. To do this, first note that since consumers have monotonic preferences, it must be the case that $p^* \cdot \left(\sum_{i \in \mathcal{I}} x_i^*\right) \leq p^* \cdot \left(\sum_{i \in \mathcal{I}} \omega_i\right)$—if we did not have to worry about $x_i \in \mathcal{T}_i$ for each i, this inequality would hold with equality by Walras's Law. Next, since $p^* \in a^*(x^*)$, we have

$$0 \geq p^* \cdot \left(\sum_{i \in \mathcal{I}} x_i^* - \sum_{i \in \mathcal{I}} \omega_i \right) \geq p \cdot \left(\sum_{i \in \mathcal{I}} x_i^* - \sum_{i \in \mathcal{I}} \omega_i \right) \text{ for all } p \in \Delta,$$

so that $\sum_{i \in \mathcal{I}} x_i^* - \sum_{i \in \mathcal{I}} \omega_i \leq 0$ and therefore $x_i^* \leq \sum_{i \in \mathcal{I}} \omega_i$ for all i, so that $x_i^* \in int(\mathcal{T}_i)$. We therefore have that $x_i^* \in \operatorname{argmax}_{x_i \in \mathcal{B}_i(p^*)} u_i(x_i)$ because if there were some $\hat{x}_i \in \mathcal{B}_i(p^*)$ with $u_i(\hat{x}_i) > u_i(x_i^*)$, then for some small λ, $\lambda \hat{x}_i + (1-\lambda) x_i^* \in \mathcal{B}_i(p^*) \cap \mathcal{T}_i$ and by (quasi-)concavity of u_i, $u_i(\lambda \hat{x}_i + (1-\lambda) x_i^*) > u_i(x_i^*)$, which is a contradiction.

We now establish that the market-clearing condition is satisfied. Since x_i^* is consumer-optimal for each i, Walras's Law tells us that $p^* \cdot \left(\sum_{i \in \mathcal{I}} x_i^*\right) = p^* \cdot \left(\sum_{i \in \mathcal{I}} \omega_i\right)$, and in particular, at p^*, the Walrasian auctioneer's value is zero (recall that the auctioneer maximizes $p \cdot \left(\sum_{i\in\mathcal{I}} x_i - \sum_{i\in\mathcal{I}} \omega_i\right)$). If $\sum_{i\in\mathcal{I}} x^*_{l,i} - \sum_{i\in\mathcal{I}} \omega_{l,i}$ were positive for any commodity l, then the auctioneer could set $p_l = 1$ and $p_{l'} = 0$ for all $l' \neq l$ and attain a positive value. This implies that no commodity is over-demanded at the allocation x^*, that is, $\sum_{i\in\mathcal{I}} x^*_i \leq \sum_{i\in\mathcal{I}} \omega_i$.

It remains only to show that this inequality actually holds with equality. By Walras's law, we know that $p^* \cdot \left(\sum_{i\in\mathcal{I}} x^*_i\right) = p^* \left(\sum_{i\in\mathcal{I}} \omega_i\right)$. Since there is no excess demand, this implies that commodity l can be in excess supply only if its price is $p^*_l = 0$. In that case, we can just modify the allocation x^* by giving the entire excess supply of commodity l to some consumer—without loss of generality, let that be consumer 1. This is feasible, and it does not affect consumer 1's utility. Why doesn't it affect her utility? Since her preferences are monotone, giving her more of commodity l cannot decrease her utility. It also cannot increase her utility, because otherwise, she would have chosen the resulting consumption bundle rather than x^*_i, and doing so would have been affordable, because $p^*_l = 0$.

To summarize, either (x^*, p^*) is a Walrasian equilibrium or the allocation resulting from arbitrarily allocating any commodity in excess supply to consumers (along with the price vector p^*) is a Walrasian equilibrium. In either case, a Walrasian equilibrium exists.■

2 Uniqueness, Stability, and Testability

We now provide an introduction to some of the most important positive properties of general equilibrium theory. We will ask when a Walrasian equilibrium is unique, whether it is stable in the sense that it can be reached by a simple price adjustment process, and we will look at whether Walrasian equilibrium imposes substantive restrictions on observable data.

This lecture will be less formal than previous lectures, mostly going through each of these topics at a rather high level. We have already alluded to the answers to some of these questions: no, Walrasian equilibria need not be unique, and no, it is not the case that a simple price adjustment process will always converge to a Walrasian equilibrium. We will first establish these results under general preferences. We will then focus on a special class of economies in which consumer preferences satisfy the *gross substitutes property*—when this property is satisfied, the model is particularly well-behaved: there will be a unique Walrasian equilibrium, and there will be a simple price-adjustment process that will always converge to it.

2.1 Uniqueness and Stability under Fairly General Preferences

Uniqueness We will first look at the question of whether there is a globally unique Walrasian equilibrium. Recall from the previous lecture the definition of the aggregate excess demand function $z(p) = \sum_{i \in \mathcal{I}} z_i(p)$, where $z_i(p) = x_i(p, p \cdot \omega_i) - \omega_i$.

Let us consider a two-commodity, two-consumer economy and normalize $p_2 = 1$. We argued informally last time that a Walrasian equilibrium exists by claiming that $z_1(p_1, 1)$ is continuous in p_1, $z_1(p_1, 1) > 0$ for p_1 small, and $z_1(p_1, 1) < 0$ for p_1 large. By the intermediate-value theorem, there exists a p_1^* such that $z_1(p_1^*, 1) = 0$ and therefore $(p_1^*, 1)$ is a Walrasian equilibrium price vector.

Is there any reason to think that there is only one p_1^* at which $z_1(p_1^*, 1) = 0$? Yes, if $z_1(p_1, 1)$ is everywhere downward-sloping, and in some sense, this is the natural case. It just says that there is less aggregate demand for commodity 1 when p_1 is higher, and we will show later that when the economy satisfies the gross substitutes condition, this will always be the case. But there certainly are situations where $z_1(p_1, 1)$ is not always downard-sloping. There is the somewhat pathological case in which commodity 1 is a Giffen good, so that $x_{1,i}(p, w)$ is increasing in p_1 even holding w fixed. Even if neither good is a Giffen good, however, $x_{1,i}(p, p \cdot \omega_1)$ may be increasing in p_1 because consumer i's wealth is increasing in p_1, so an upward-sloping region of $z_1(p_1, 1)$ is not particularly implausible.

In the first lecture, we discussed an example in the Edgeworth box in which the two consumers' offer curves intersected at three equilibrium points, and in the first problem set, you were asked to solve for the set of Walrasian equilibria in a numerical example for which there were three equilibria. Recall the example from the problem set. Consumers' preferences and endowments are:

$$\begin{aligned} u_1(x_{1,1}, x_{2,1}) &= \left(x_{1,1}^{-2} + \left(\frac{12}{37}\right)^3 x_{2,1}^{-2}\right)^{-1/2}, & \omega_1 = (1,0), \\ u_2(x_{1,2}, x_{2,2}) &= \left(\left(\frac{12}{37}\right)^3 x_{1,2}^{-2} + x_{2,2}^{-2}\right)^{-1/2}, & \omega_2 = (0,1). \end{aligned}$$

If we normalize $p_2 = 1$, consumers' Marshallian demands for commodity 1 are:

$$x_{1,1}((p_1, 1), p_1) = \frac{p_1}{p_1 + \frac{12}{37} p_1^{1/3}}, \; x_{1,2}((p_1, 1), 1) = \frac{1}{p_1 + \frac{37}{12} p_1^{1/3}},$$

and the aggregate excess demand for commodity 1 is therefore

$$z_1(p_1, 1) = \frac{1}{p_1 + \frac{37}{12} p_1^{1/3}} - \frac{\frac{12}{37} p_1^{1/3}}{p_1 + \frac{12}{37} p_1^{1/3}}.$$

Figure 11 plots $z_1(p_1, 1)$ and shows that there are three solutions to $z_1(p_1, 1) = 0$. You might recall from the first problem set that for $p_1^* \in \left\{\frac{27}{64}, 1, \frac{64}{27}\right\}$, there is a Walrasian equilibrium with prices $(p_1^*, 1)$.

There are two additional general points that you can see illustrated in both Figures 5(a) and 11. The first is that if you were to perturb the economy slightly by changing consumers' preferences or endowments by a tiny amount, this would not affect the fact that there are three Walrasian equilibria.

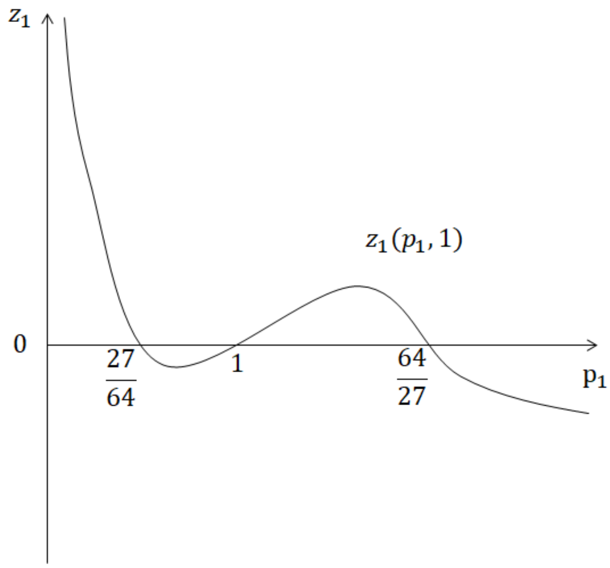

Figure 11: Multiple Walrasian equilibria

The second general point that these examples illustrate is that even though Walrasian equilibria may not be globally unique, they may be what is referred to as *locally unique* in the sense that there is no other Walrasian equilibrium price vector within a small enough range around the original equilibrium price vector. Figure 12 illustrates an example for which this is not the case. An equilibrium is not locally unique if its price vector p is the limit of a sequence of other equilibrium price vectors. This example shows that this can happen, but only if $z_1(p_1, 1)$ is flat and equal to zero over some interval of prices $[p_1^*, p_1^{**}]$. The important point to note about this example is that it is not *generic*: any small perturbation of $z_1(\cdot, 1)$ that would arise from, say, a change in endowments, would restore the property that there are a finite number of equilibria.

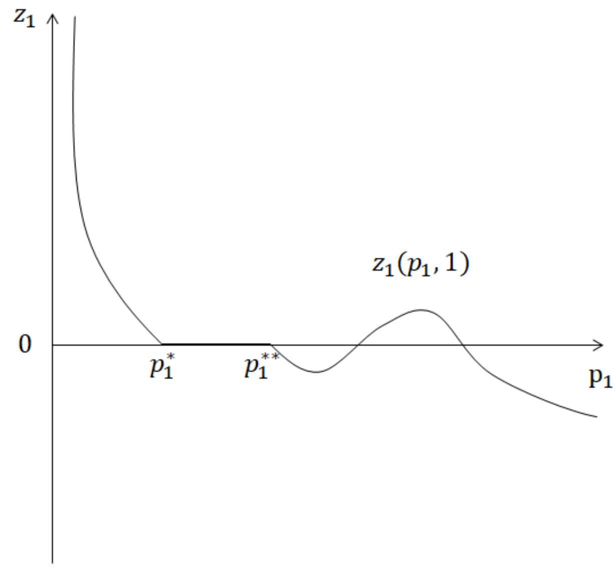

Figure 12: Walrasian Equilibria need not be locally unique

Exercise 11 (Adapted from MWG 17.D.1). Consider an exchange economy with two commodities and two consumers. Both consumers have homothetic preferences of the constant elasticity variety. Moreover, the elasticity of substitution is the same for both consumers and is small (i.e., commodities are close to perfect complements). Specifically,

$$u_1(x_{1,1}, x_{2,1}) = \left(2x_{1,1}^{\rho} + x_{2,1}^{\rho}\right)^{1/\rho} \text{ and } u_2(x_{1,2}, x_{2,2}) = \left(x_{1,2}^{\rho} + 2x_{2,2}^{\rho}\right)^{1/\rho},$$

and $\rho = -4$. The endowments are $\omega_1 = (1, 0)$ and $\omega_2 = (0, 1)$. Compute the excess demand function of this economy and find the set of competitive equilibria.

Tatonnement Stability One important aspect of Walrasian equilibrium that we have alluded to throughout the course but have not yet addressed is: where do Walrasian equilibrium prices come from? General equilibrium theory is quite weak on the kinds of price-adjustment processes that might lead to Walrasian equilibrium outcomes.

Walras proposed a process he called "tatonnement" whereby a fictitious Walrasian auctioneer gradually raises the price of commodities in excess demand and reduces the prices of those in excess supply until markets clear. This process is related to what the Walrasian auctioneer did in our proof of existence from last time, but not quite the same. In particular, the process last time adjusted prices discontinuously, but it was aimed at showing the existence of a fixed point for a particular operator, not at showing that the fixed point(s) of that operator could be found by iterating it from an arbitrary starting point.

Formally, consider the following continuous-time price-adjustment process $p(t)$:

$$\frac{dp(t)}{dt} = \alpha z(p(t)),$$

for some constant $\alpha > 0$. Given a starting price vector $p(0)$, the process raises prices for any commodities l for which $z_l(p(t)) > 0$ (i.e., for which there is excess demand), and it reduces prices for those for which $z_l(p(t)) < 0$.

The stationary points of this process are prices p at which $z(p) = 0$: Walrasian equilibrium prices. An equilibrium price vector p^* is said to be *locally stable* if the price-adjustment process converges to p^* from any "nearby" price vectors, and it is *globally stable* if the process converges to p^* from *any* initial starting price vector. Does this process converge to a Walrasian equilibrium price vector? When there are only two commodities, and the economy satisfies properties $(A1)$, $(A2')$, $(A3)$, and $(A4)$, this process does in fact converge, as Figure 13 highlights. Here, we can also see that p_1^* and p_1^{***} are locally stable, and p_1^{**} is not, and none of the equilibrium price vectors is globally stable.

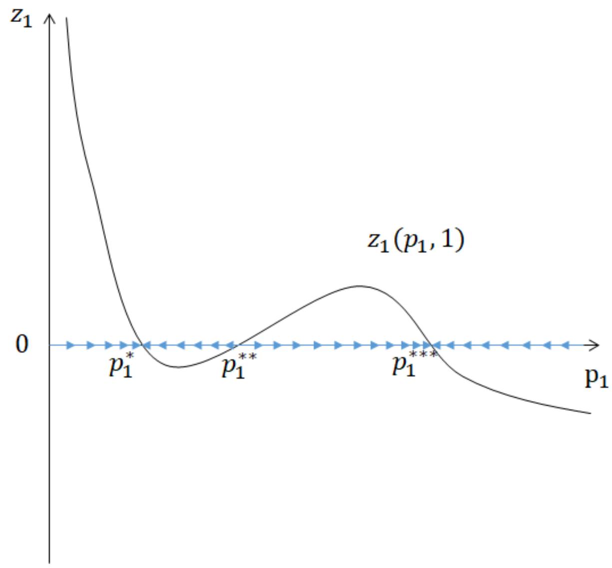

Figure 13: Tatonnement process for two commodities

This price-adjustment process gives us a way to study how equilibrium prices might be reached, but it has several drawbacks. First, the process itself is a conceptual exercise rather than a practical one—the GE model predicts that no one will trade at non-equilibrium prices. Second, if one were to try to implement this process by asking consumers how much they would demand at different price levels, then they would be unlikely to want to report their demands truthfully. Finally, the main drawback with this procedure is that it does not in general converge to an equilibrium price vector. In a famous paper, Scarf (1960) provided several examples in which the process does not converge when there are more than two commodities. We will show in the next section, however, that there are classes of economies for which it does converge.

2.2 Uniqueness and Stability under Gross Substitutes

In this section, we will show that economies that satisfy the gross substitutes property have particularly nice properties: there is a unique Walrasian equilibrium (up to a normalization), it is globally stable, and it has nice comparative statics properties.

Recall from consumer theory that commodities k and l are gross substitutes if an increase in p_k increases the Marshallian demand for commodity l (and vice versa), holding wealth fixed. The analogous definition in general equilibrium is as follows.

Definition 4. A Marshallian demand function $x(p, p\cdot\omega)$ satisfies **gross substitutes** at endowment ω if, for all prices p and p' with $p'_k > p_k$ and $p'_l = p_l$ for all $l \neq k$, we have $x_l(p', p'\cdot\omega) > x_l(p, p\cdot\omega)$ for all $l \neq k$.

This definition of gross substitutes is more subtle than the definition you saw from consumer theory, since increasing p_k also increases the consumer's wealth. It is straightforward to show that if all commodities are gross substitutes in the consumer-theory sense and they are also normal goods (so that demand increases with wealth), then demand functions will satisfy the gross substitutes property for all possible (non-negative) endowments. It is not readily apparent from the definition of gross substitutes that the demand for commodity l is decreasing in p_l, but it is true: since demand is homogeneous of degree 0 in p, increasing p_l is the same as holding p_l fixed and decreasing all other prices. Since decreasing each of these other prices decreases demand for commodity l, so does decreasing all of them.

If each consumer i's demand function satisfies gross substitutes at ω_i, then so does aggregate demand $\sum_{i\in\mathcal{I}} x_i(p, p\cdot\omega_i)$. The property is restrictive, but it is satisfied by many common functional forms such as CES preferences: $u_i(x_i) = \left(\sum_{l\in\mathcal{L}} \alpha_l x_{l,i}^{\rho}\right)^{1/\rho}$ for $0 < \rho < 1$.

If aggregate demand satisfies the gross substitutes property, then there is a unique Walrasian equilibrium, as the following result shows.

Proposition 1. If the aggregate excess demand function $z(\cdot)$ satisfies gross substitutes, the economy has at most one Walrasian equilibrium (up to a normalization).

Proof of Proposition 1. We need to show that there is at most one (normalized) price vector p such that $z(p) = 0$. To see why this is the case, suppose $z(p) = z(p') = 0$ for two price vectors p and p' that are not collinear. By homogeneity of degree zero, we can normalize the price vectors in such a way that $p'_l \geq p_l$ for all $l \in \mathcal{L}$ and $p'_k = p_k$ for some commodity k. Then, to move from p to p', we can think about doing this in $L - 1$ steps, increasing the prices of each commodity $l \neq k$ in turn. At each step where a component of the price vector increases strictly, the aggregate demand for commodity k must strictly increase, so that $z_k(p') > z_k(p) = 0$. Moreover, there must be at least one such k, since p is not collinear with p', yielding a contradiction.■

When aggregate demand satisfies the gross substitutes property, not only is there a unique Walrasian equilibrium, but the tatonnement price-adjustment process we described above globally converges to it. To establish this result, we will first prove a lemma.

Lemma 4. Suppose that the aggregate excess demand function $z(\cdot)$ satisfies gross substitutes and that $z(p^*) = 0$. Then for any p not collinear with p^*, $p^* \cdot z(p) > 0$.

Proof of Lemma 4. We will give the proof in the $L = 2$ case. Normalize $p_2 = p_2^* = 1$. Then

$$\begin{aligned} p^* \cdot z(p) &= (p^* - p) \cdot (z(p) - z(p^*)) \\ &= (p_1^* - p_1)(z_1(p) - z_1(p^*)) > 0. \end{aligned}$$

The first equality uses Walras's Law (giving us that $p \cdot z(p) = 0$) and the fact that p^* is a Walrasian equilibrium (so that $z(p^*) = 0$). The second equality uses the normalization $p_2 = p_2^* = 1$. The inequality follows from the gross substitutes property: $p_1 > p_1^*$ implies $z_1(p) < z_1(p^*)$ and $p_1 < p_1^*$ implies $z_1(p) > z_1(p^*)$.■

With this Lemma, we can prove that the tatonnement process converges to the unique (up to normalization) Walrasian equilibrium price vector p^*.

Proposition 2. Suppose that the aggregate excess demand function $z(\cdot)$ is satisfies the gross substitutes property and that p^* is a Walrasian equilibrium price vector. Then the price-adjustment process $p(t)$ defined by $dp(t)/dt = \alpha z(p(t))$, with $\alpha > 0$, converges to p^* for any initial condition $p(0)$.

Proof of Proposition 2. To prove this result, we will show that the squared distance between $p(t)$ and p^* decreases monotonically in t. Let $D(p) = \frac{1}{2}\sum_{l\in\mathcal{L}}(p_l - p_l^*)^2$ denote the distance between p and p^*. Then

$$\begin{aligned}
\frac{dD(p(t))}{dt} &= \sum_{l\in\mathcal{L}}(p_l(t) - p_l^*)\frac{dp_l(t)}{dt} \\
&= \alpha\sum_{l\in\mathcal{L}}(p_l(t) - p_l^*)\, z_l(p(t)) \\
&= -\alpha p^* \cdot z(p) \leq 0,
\end{aligned}$$

where the third equality uses Walras's law. By the previous lemma, the last inequality is strict unless p is collinear with p^*. Since $D(p(t))$ is monotonic and bounded, it must converge to some value $\delta \geq 0$. If $\delta = 0$, we are done. If $\delta > 0$, then there is a contradiction, because continuity of aggregate demand implies that $p^* \cdot z(p(t))$ is bounded away from 0 for all $p(t)$ bounded away from p^*.■

Finally, economies with the gross substitutes property have nice comparative statics. Any change that raises excess demand for commodity k will increase its equilibrium price. As an example, suppose there are two commodities and normalize $p_2 = 1$. Suppose also that commodity 1 is a normal good for all consumers. Consider an increase in the aggregate endowment for commodity 2. For any price p_1, this will increase aggregate demand for commodity 1 and hence increase $z_1(\cdot, 1)$.

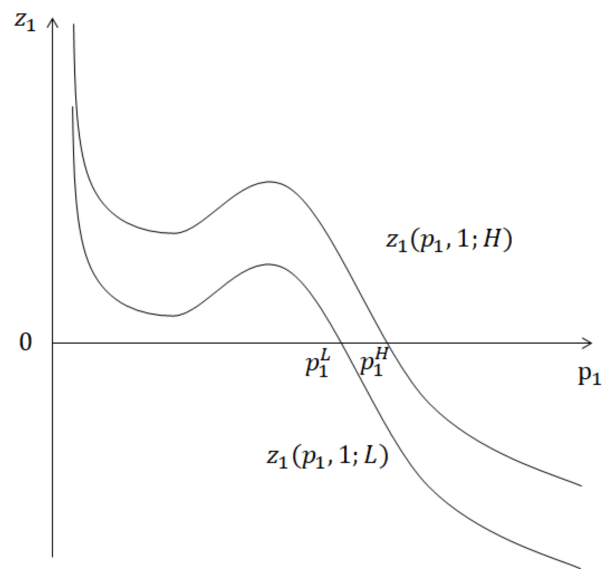

Figure 14: Comparative statics

Figure 14 compares the aggregate excess demand functions for two economies: one $(z_1(\cdot, 1; L))$ with a low aggregate endowment of commodity 2 and one $(z_1(\cdot, 1; H))$ with a high aggregate endowment of commodity 2. The curve $z_1(\cdot, 1; H)$ lies above $z_1(\cdot, 1; L)$ and because it is continuous and crosses zero once, the new equilibrium price vector must have a higher price for commodity 1.

2.3 Empirical Content of GE

As we just saw, whether there is a unique Walrasian equilibrium and whether Walrasian equilibria are stable depended critically on the structure of the economy's aggregate excess demand function $z(\cdot)$.

What do we know in general about the structure of aggregate excess demand? We proved that under assumptions $(A1)$, $(A2')$, $(A3)$, and $(A4)$ about consumer preferences and endowments that $z(\cdot)$ is continuous, homogeneous of degree zero in p, it satisfies Walras's Law, and $\lim_{p \to 0} z(p) \to \infty$. But, as Sonnenschein (1973) showed for the case of two commodities, and Mantel (1974) and Debreu (1974) showed more generally, the assumption of consumer maximization alone imposes no further restrictions on $z(\cdot)$. This is a very negative result, since it implies that even if we observe an economy in a Walrasian equilibrium with price vector p, it is possible for the same economy to have an arbitrary number of Walrasian equilibria with arbitrary stability properties.

Theorem 5 (Sonnenschein-Mantel-Debreu Theorem). For any closed and bounded set of positive prices $\mathcal{P} \subseteq \mathbb{R}^L_{++}$ and any function $f : \mathcal{P} \to \mathbb{R}^L$ satisfying continuity, homogeneity of degree 0, and Walras's Law, there exists an exchange economy with L consumers with continuous, strictly convex, and monotone preferences whose aggregate excess demand function coincides with f on $\mathcal{P}$.

We omit the proof here. See MWG Chapter 17.E for a proof in the $L = 2$ case and a discussion about the more general proof. Roughly speaking, the structure of the proof begins with a candidate excess demand function $f(p)$ that is continuous, homogeneous of degree 0, and satisfies Walras's Law and reverse engineers a set of consumer preferences and endowments that generate $f(p)$ as the aggregate excess demand function. The ability to do so requires a lot of flexibility in specifying consumer preferences that feature potentially strong income effects as well as the ability to specify consumers' endowments. A common interpretation of this theorem is that "anything goes" in general equilibrium theory. That is, without making strong assumptions on preferences: (i) pretty much any comparative statics result could be obtained in a general equilibrium model, and (ii) general equilibrium theory has essentially no empirical content. This is not quite right, though, as we will now see.

Brown and Matzkin (1996) prove an important result showing that if an economist is able to observe endowments as well as prices, then the Walrasian model is in principle testable. That is, there are endowment and price pairs $\left(p, (\omega_i)_{i\in\mathcal{I}}\right)$ and $\left(p', (\omega_i')_{i\in\mathcal{I}}\right)$ such that if p is a Walrasian equilibrium price vector given a fixed set of consumers with endowments $(\omega_i)_{i\in\mathcal{I}}$, then if the same set of consumers instead had endowments $(\omega_i')_{i\in\mathcal{I}}$, p' could not be a Walrasian equilibrium price vector.

Theorem 6 (Brown-Matzkin Theorem). There exists price-endowment pairs $\left(p, (\omega_i)_{i\in\mathcal{I}}\right)$ and $\left(p', (\omega'_i)_{i\in\mathcal{I}}\right)$ such that there do not exist monotone preferences $(u_i)_{i\in\mathcal{I}}$ such that p is a Walrasian equilibrium price vector for the exchange economy $(u_i, \omega_i)_{i\in\mathcal{I}}$ and p' is a Walrasian equilibrium price vector for the exchange economy $(u_i, \omega'_i)_{i\in\mathcal{I}}$.

Proof of Theorem 6. We can prove this theorem in the case of two consumers and two commodities. Consider the two Edgeworth boxes in Figure 15. Because p is a Walrasian equilibrium price vector given endowment ω, consumer 1 must weakly prefer some bundle on the segment A to any bundle on the segment B. By monotonicity, for every point on the segment A', there is some point on B that consumer 1 strictly prefers. There is therefore some bundle on A that is preferred by consumer 1 to every bundle on A'. If p' is a Walrasian equilibrium price vector given ω', we have a contradiction: every bundle on A is available to consumer 1 at prices p', yet she chooses a bundle on A'.■

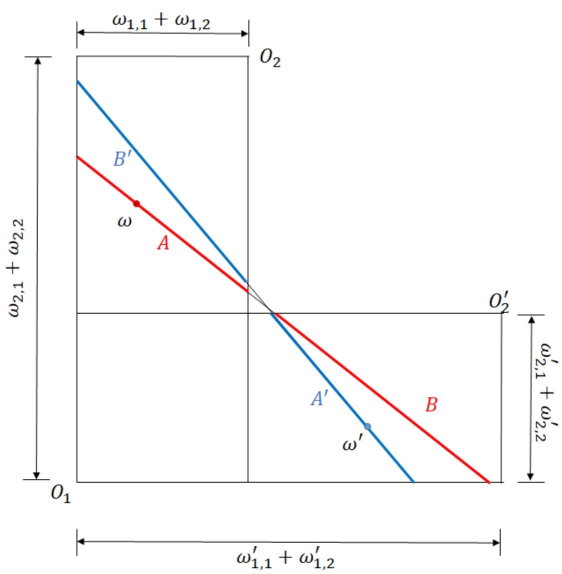

Figure 15: Brown-Matzkin theorem

The Brown-Matzkin theorem shows that, in order to construct the arbitrary excess demand functions that the proof of the Sonnenschein-Mantel-Debreu theorem requires, you really need the flexibility in specifying arbitrary endowments in addition to flexibility in specifying preferences. It also illustrates a more general point that, even if at its highest level, a theory imposes little structure on endogenous variables, imposing more structure on the theory typically imposes more structure on its implications.