Economics 2010b: Microeconomic Theory II

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Introduction; GE Model; Welfare Theorems; Pareto optimality

1 Introduction to General Equilibrium Theory[1]

In the first three weeks of this course, our goal is to develop a parsimonious model of the overall economy to study the interaction of individual consumers and firms in perfectly competitive *decentralized* markets. The resulting framework has provided the workhorse micro-foundations for much of modern macroeconomics, international trade, and financial economics. In the last three and a half weeks, we will begin to study *managed* transactions and in particular how individuals should design institutions such as contracts and property rights allocations to governance structures, to achieve desirable outcomes.

The main ideas of general equilibrium theory have a long history, going back to Adam Smith's (1776) evocative descriptions of how competition channels individual self-interest in the social interest and how a sense of "coherence among the vast numbers of individuals and seemingly separate decisions" (Arrow, 1972) can arise in the economy without explicit design. General equilibrium theory addresses how this aggregate "coherence" emerges from individual interactions and can potentially lead to socially desirable allocations of goods and services in the economy. The mechanism through which this coherence emerges is, of course, the price mechanism. Individuals facing the same, suitably determined, prices will end up making decisions that are well-coordinated at the economy-wide level.

What distinguishes general equilibrium theory from partial equilibrium theory, which you have studied in Economics 2010a, is the idea that if we want to develop a theory of the price system for the economy as a whole, we have to consider the equilibrium in all markets in the economy simultaneously. As you can imagine, thinking about all markets simultaneously can be a complicated endeavor, since markets are interdependent: the price of computer chips

[1] These notes borrow liberally from Levin's (2006) notes and Wolitzky's (2016) notes.

will affect the price of software, cars, appliances, and so on. General equilibrium theory was the most active research area in economic theory for a good part of the 20th century and is therefore a very rich topic. Our goal over the first three weeks of the course is to cover only the basics.

In many ways, the first part of this class will be structured the way an applied theory paper is structured. We will start by talking about the setup of a model: who are the players, what do they do, what do they know, what are their preferences, what is the solution concept we will be using? Then we will partially characterize its solution, focusing on its efficiency properties in particular. What do I mean by partially characterize? I mean that we will be describing some properties of equilibria that we can talk about without actually solving the full model. That should take us through the first week.

In the second week, we will do a little bit of heavier lifting and begin with a question that is often easy to overlook but is very important to answer: does an equilibrium exist? Existence proofs can sometimes seem like a bit of an esoteric detour, but a good existence proof is especially useful if it helps build tools to answer other important questions about the model. After we establish that an equilibrium exists, we will ask questions like: when is there a unique equilibrium? Are equilibria stable? What are the testable implications of equilibrium behavior?

The third week will focus on what I will blandly call "extensions," but are really the meat and potatoes of how general equilibrium theory gets used in practice. We will show how we can introduce firms, time, and uncertainty into the framework, and we will talk about how and when the main results we identified above apply in these settings as well. And finally, we will end by putting the model's solution concept, competitive equilibrium, on firmer microfoundations. A lot of economics is buried in the solution concept, and developing microfoundations for the solution concept is a useful way to flesh out some of the key insights.

2 Pure Exchange Economies

A general equilibrium model describes three basic activities that take place in the economy: production, exchange, and consumption. For the first two weeks, we will set production aside and focus on the minimal number of modeling ingredients necessary to give you a flavor of the powerful results of general equilibrium theory.

We begin with a description of the model we will be using. As always, the exposition of an economic model specifies a complete description of the economic environment (players, actions, and preferences) and the solution concept that will be used to derive prescriptions and predictions.

2.1 The Model

Formally, a pure exchange economy is an economy in which there are no production opportunities. There are I **consumers** $i \in \mathcal{I} = \{1, \ldots, I\}$ who buy, sell, and consume L **commodities**, $l \in \mathcal{L} = \{1, \ldots, L\}$. A **consumption bundle for consumer** i is a vector $x_i = (x_{1,i}, \ldots, x_{L,i})$, where $x_i \in \mathcal{X}_i$, which is consumer i's **consumption set** and just describes her feasible consumption bundles. We will assume throughout that $\mathcal{X}_i$ contains the 0 vector and is a convex set. Consumer i has an **endowment** of the L commodities, which is described by a vector $\omega_i = (\omega_{1,i}, \ldots, \omega_{L,i})$ and has preferences over consumption bundles, which we assume can be represented by a utility function $u_i : \mathcal{X}_i \to \mathbb{R}$. A **pure exchange economy** is therefore a set $\mathcal{E} = \left((u_i, \omega_i)_{i \in \mathcal{I}}\right)$, which fully describes the model's primitives: the set of players, their preferences, and their endowments.

Each consumer takes **prices** $p = (p_1, \ldots, p_L)$ as given and solve her **consumer maximization problem**:

$$\max_{x_i \in \mathcal{X}_i} u_i(x_i) \text{ s.t. } p \cdot x_i \leq p \cdot \omega_i,$$

where the right-hand side of the consumer's budget constraint is her wealth, as measured by the market value of her endowment at prices p. Consumer i's feasible actions are therefore

$x_i \in \mathcal{B}_i(p) \equiv \{x_i \in \mathcal{X}_i : p \cdot x_i \leq p \cdot \omega_i\}$, where we refer to $\mathcal{B}_i(p)$ as her **budget set at prices** p. Given prices p and endowment ω_i, we will refer to consumer i's optimal choices as her **Marshallian demand correspondence** and denote it by $x_i(p, p \cdot \omega_i)$. Simply put, all consumers do in this model is to choose their favorite consumption bundles in their budget sets.

We have now described the players and their actions, but no model description is complete without a solution concept. Here, the solution concept will be a Walrasian equilibrium, which will specify a set of prices and a consumption bundle for each consumer (we will refer to a collection of consumption bundles for each consumer as an **allocation**) that satisfy two properties: consumer optimization and market-clearing. Given prices p, each consumer optimally chooses her consumption bundle, and total demand for each commodity equals total supply.

Definition 1. A **Walrasian equilibrium** for the pure exchange economy $\mathcal{E}$ is a vector $\left(p^*, (x_i^*)_{i \in \mathcal{I}}\right)$ that satisfies:

1. Consumer optimization: for all consumers $i \in \mathcal{I}$,

$$x_i^* \in \operatorname{argmax}_{x_i \in \mathcal{B}_i(p^*)} u_i(x_i),$$

2. Market-clearing: for all commodities $l \in \mathcal{L}$,

$$\sum_{i \in \mathcal{I}} x_{l,i}^* = \sum_{i \in \mathcal{I}} \omega_{l,i}.$$

We have now fully specified the model, but before we start to go into more detail discussing the properties of Walrasian equilibria, there are a couple more important definitions to introduce. The first is the notion of a feasible allocation, which is just a collection of consumption bundles for each consumer for which the total amount consumed for each commodity does not exceed the total endowment of that commodity.

Definition 2. An allocation $(x_i)_{i\in\mathcal{I}} \in \mathbb{R}_+^{L\cdot I}$ is **feasible** if for all $l \in \mathcal{L}$, $\sum_{i\in\mathcal{I}} x_{l,i} \leq \sum_{i\in\mathcal{I}} \omega_{l,i}$.

The next definition is going to describe how we will be thinking about optimality in the economy. For a lot of optimization problems you have seen in your other courses, the appropriate notion of optimality is straightforward. For example, if a consumer has a well-defined utility function, it is straightforward to think about what is optimal for her given her budget set. Once we start thinking about environments with more than one consumer, we would, in some sense, like to maximize multiple objective functions (i.e., each consumer's utility) simultaneously. In general, there are no allocations that simultaneously maximize the utility of all consumers—consumers' objectives are typically in conflict with one another's—so the appropriate notion of optimality is not as straightforward. The notion will we use is that of Pareto optimality, which means that all we are doing is ruling out allocations that are dominated by other feasible allocations.

Definition 3. Given an economy $\mathcal{E}$, a feasible allocation $(x_i)_{i\in\mathcal{I}}$ is **Pareto optimal (or Pareto efficient)** if there is no other feasible allocation $(\hat{x}_i)_{i\in\mathcal{I}}$ such that $u_i(\hat{x}_i) \geq u_i(x_i)$ for all $i \in \mathcal{I}$ with strict inequality for some $i \in \mathcal{I}$.

In words, all Pareto optimality rules out is allocations for which someone could be made better off without making anyone else worse off. This notion of optimality is therefore silent on issues of distribution, since it may be Pareto optimal for one consumer to consume everything in the economy and for everyone else to consume nothing.

2.2 Assumptions on Consumer Preferences and Endowments

Throughout the next few sections, we will invoke different sets of assumptions for different results. I will collect these assumptions here and will be explicit in referring to them when they are required for a result.

Assumption A1 (continuity): For all consumers $i \in \mathcal{I}$, u_i is continuous.

Assumption A2 (monotonicity): For all consumers $i \in \mathcal{I}$, u_i is increasing: $u_i(x_i') >$

$u_i(x_i)$ whenever $x'_{l,i} > x_{l,i}$ for all $l \in \mathcal{L}$.

Assumption A3 (concavity): For all consumers $i \in \mathcal{I}$, u_i is concave.

Assumption A4 (interior endowments): For all consumers $i \in \mathcal{I}$, $\omega_{l,i} > 0$ for all $l \in \mathcal{L}$.

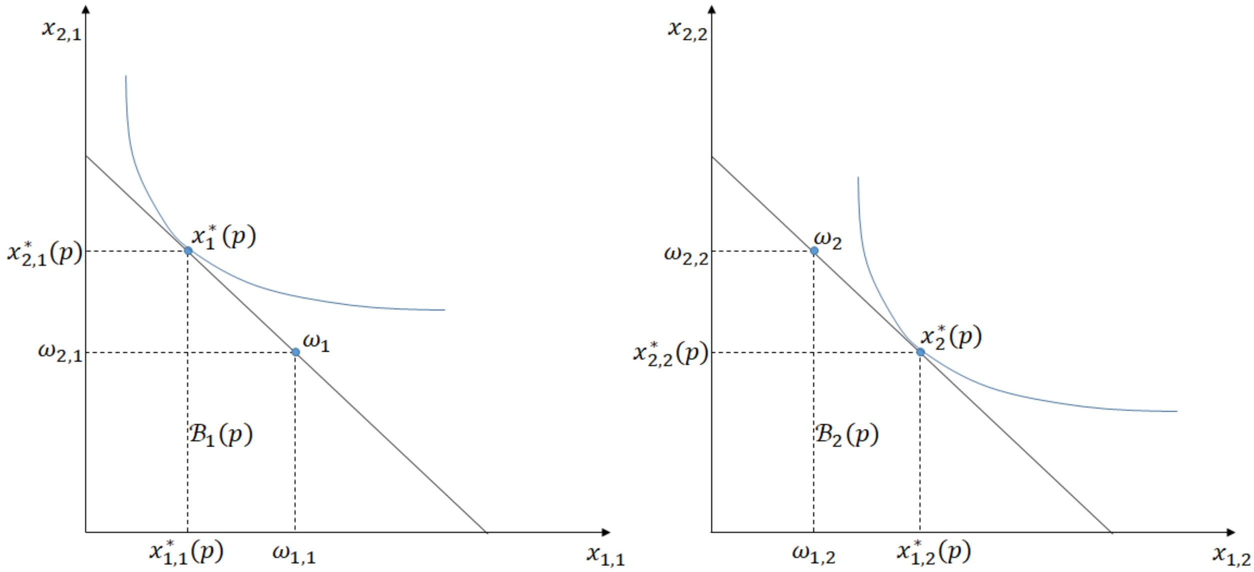
The first three assumptions should be familiar from Economics 2010a. The results we will be establishing in the upcoming sections will hold under weaker assumptions—for example, $(A2)$ can typically be relaxed to local nonsatiation,[2] and $(A3)$ can typically be relaxed to quasiconcavity. The last assumption is a strong assumption that will prove to be sufficient for ruling out some pathological cases in which a Walrasian equilibrium does not exist.

2.3 Graphical Examples

Many of the main ideas of general equilibrium theory can be understood in a two-consumer, two-commodity pure exchange economy. We can get most of the results across graphically in what is referred to as an Edgeworth box. Edgeworth boxes are informationally dense, so let me introduce the constituent elements separately.

Figure 1(a) depicts the relevant information for consumer 1. On the horizontal axis is her consumption of commodity 1 and on the vertical axis is her consumption of commodity 2. Her endowment is $\omega_1 = (\omega_{1,1}, \omega_{2,1})$. At prices $p = (p_1, p_2)$, she can afford to buy any consumption bundle in the set $\mathcal{B}_1(p)$. The slope of her budget line is $-p_1/p_2$. Given her preferences, which are represented by her indifference curve, and given prices p and endowment ω_1, she optimally chooses to consume $x_1^*(p) = \left(x_{1,1}^*(p), x_{2,1}^*(p)\right)$. In other words, at prices p, she would optimally like to sell $\omega_{1,1} - x_{1,1}^*(p)$ units of commodity 1 in exchange for $x_{2,1}^*(p) - \omega_{2,1}$ units of commodity 2. Figure 1(b) depicts the same information for consumer 2.

[2] We say that consumer i's preferences satisfy **local non-satiation** if for every $x_i \in \mathcal{X}_i$ and every $\varepsilon > 0$, there is an $x'_i \in \mathcal{X}_i$ such that $||x'_i - x_i|| \leq \varepsilon$ and $u_i(x'_i) > u_i(x_i)$.



Figures 1(a) and 1(b): consumer-optimization problems

The Edgeworth box represents both consumers' endowments and their optimal choices as a function of the prices p, so it will incorporate all the information in Figures 1(a) and 1(b). To build towards this goal, Figure 2(a) depicts all the **non-wasteful allocations** in the economy: allocations $(x_i)_{i\in\{1,2\}}$ for which $x_{l,1} + x_{l,2} = \omega_{l,1} + \omega_{l,2}$ for $l \in \{1, 2\}$. The bottom-left corner is the origin for consumer 1, and the upper-right corner is the origin for consumer 2. The length of the horizontal axis is equal to the total endowment of commodity 1, and the length of the vertical axis is equal to the total endowment of commodity 2. The horizontal axis, read from the left to the right, represents consumer 1's consumption of commodity 1, and read from the right to the left, represents consumer 2's consumption of commodity 1. The vertical axis, read from the bottom to the top, represents consumer 1's consumption of commodity 2, and read from the top to the bottom, represents consumer 2's consumption of commodity 2. The endowment ω is a point in the Edgeworth box, and it represents a non-wasteful allocation, since $\omega_{l,1} + \omega_{l,2} = \omega_{l,1} + \omega_{l,2}$ for $l \in \{1, 2\}$. The allocation x also represents a non-wasteful allocation, since $x_{l,1} + x_{l,2} = \omega_{l,1} + \omega_{l,2}$ for $l \in \{1, 2\}$.

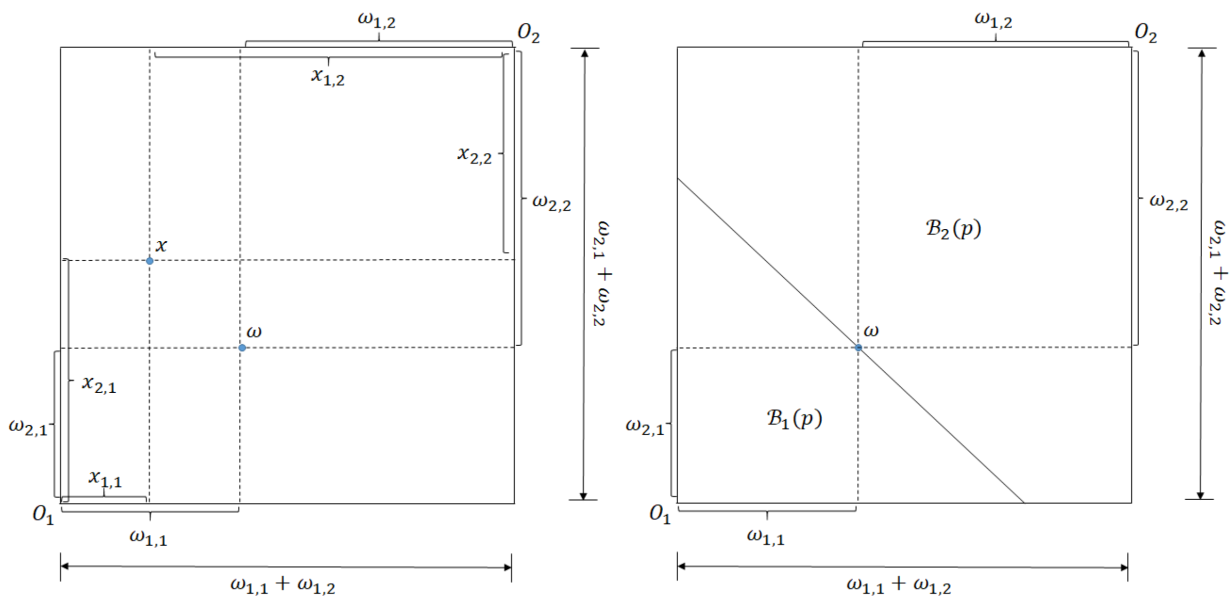
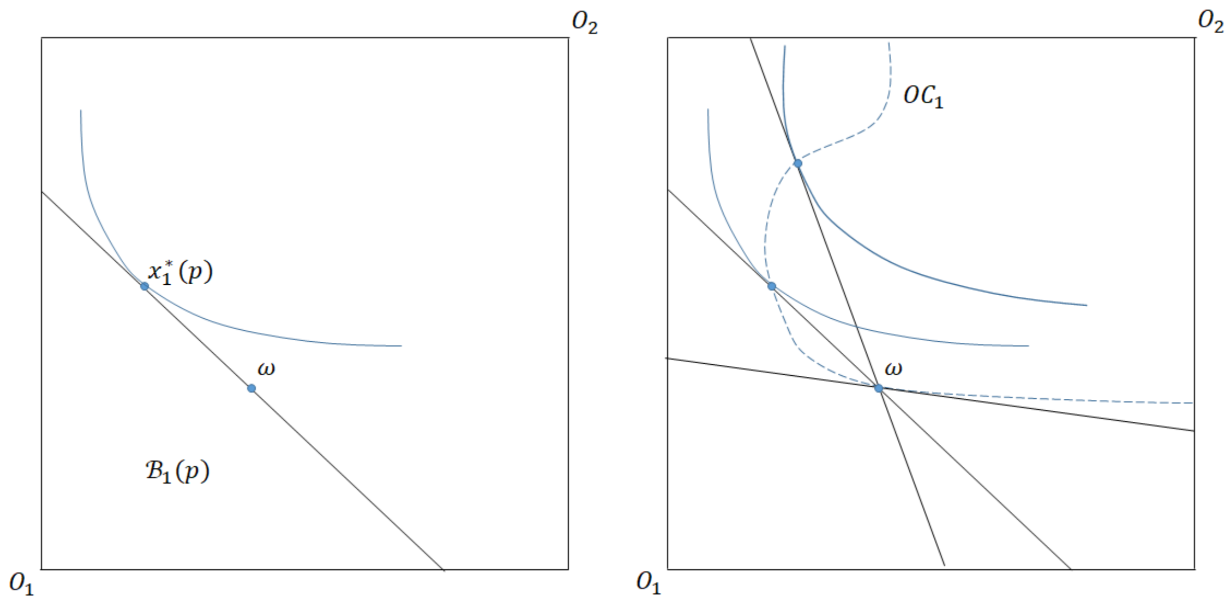


Figure 2(a): non-wasteful allocations

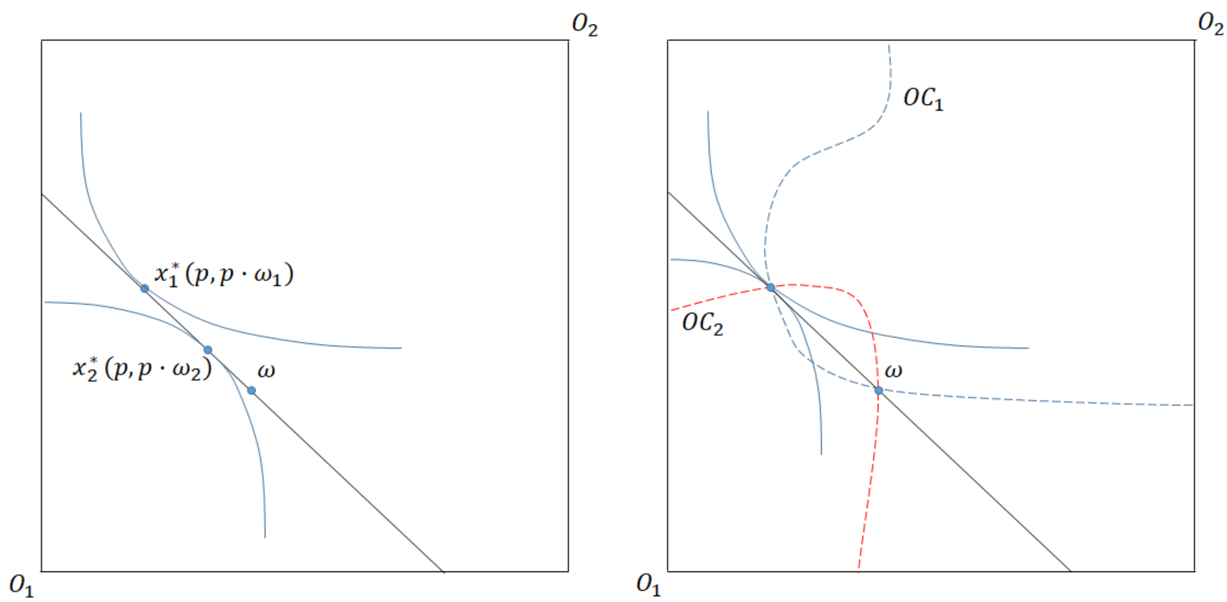
Figure 2(b): prices and budget sets

Figure 2(b) adds prices into the picture and shows that, given any price vector p, the Edgeworth box can be partitioned into the budget sets for the two consumers. Given these prices, consumer 1 can choose any consumption bundle to the bottom left of the diagonal line, and consumer 2 can choose any consumption bundle to the upper right of the diagonal line. Given these prices, Figure 3(a) shows that consumer 1 will optimally choose bundle $x_1^* (p, p \cdot \omega_1)$, and Figure 3(a) shows how $x_1^* (p, p \cdot \omega_1)$ varies as the price ratio varies. Note that, in terms of determining consumer 1's optimal choice, the price ratio p_1/p_2 is a sufficient statistic for the price vector p. This is because Marshallian demand correspondences are homogeneous of degree zero in prices (i.e., $x_1^* (p, p \cdot \omega_1) = x_1^* (\lambda p, \lambda p \cdot \omega_1)$ for all $\lambda \in \mathbb{R}_{++}$).The curve traced out in Figure 3(b) is referred to as consumer 1's **offer curve**.



Figures 3(a) and 3(b): Consumer 1's Marshallian demand for a fixed p and her offer curve

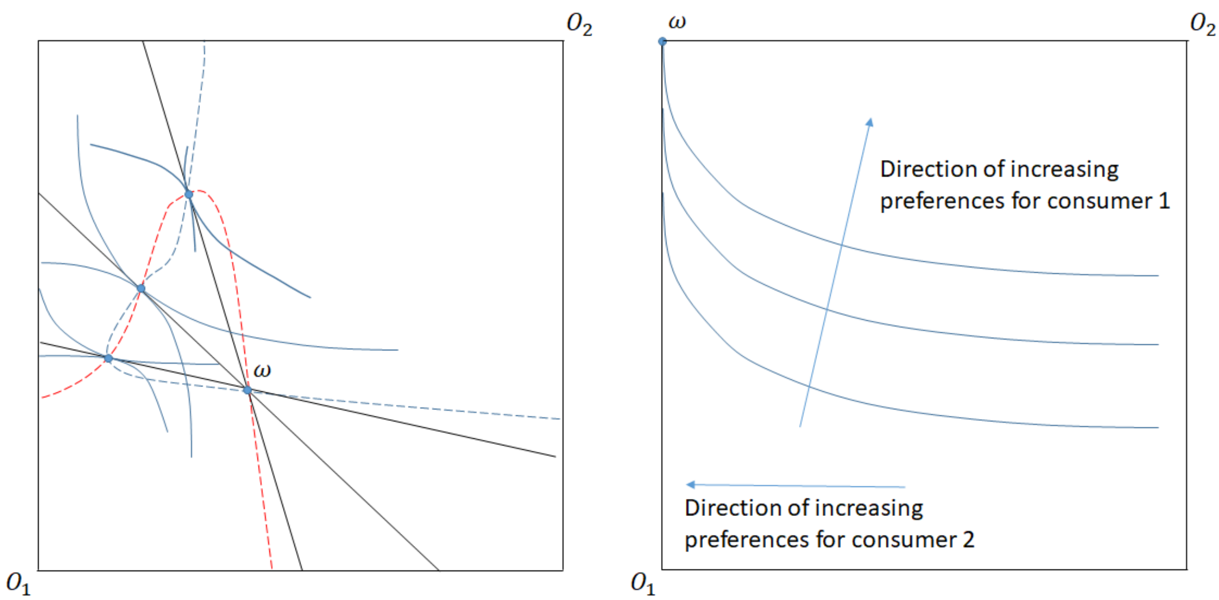
Recall from above that in a Walrasian equilibrium $\left(p^*, (x_i^*)_{i\in\{1,2\}}\right)$, consumer i optimally chooses x_i^* given equilibrium prices p^*. This means that in any Walrasian equilibrium, both consumers' optimal choices lie on their offer curves. Figure 4(a) depicts, for a given price vector p, both consumers' optimal choices. At this price vector, consumer 1 would like to sell $\omega_{1,1} - x_{1,1}^*(p, p\cdot\omega_1)$ units of commodity 1 in exchange for $x_{2,1}^*(p, p\cdot\omega_1) - \omega_{2,1}$ units of commodity 2, and consumer 2 would like to buy $x_{1,2}^*(p, p\cdot\omega_2) - \omega_{1,2}$ units of commodity 1 and sell $\omega_{2,2} - x_{2,2}^*(p, p\cdot\omega_2)$ units of commodity 2. The associated allocation, $(x_1^*(p, p\cdot\omega_1), x_2^*(p, p\cdot\omega_2))$, is not a Walrasian equilibrium allocation, since consumer 1 would like to sell more units of commodity 1 than consumer 2 would like to buy, so the market for commodity 1 does not clear: $\omega_{1,1} - x^*_{1,1}(p, p \cdot \omega_1) > x^*_{1,2}(p, p \cdot \omega_2) - \omega_{1,2}$.



Figures 4(a) and 4(b): disequilibrium and equilibrium allocations

It should be clear from the above argument, then, that any Walrasian equilibrium allocation has to occur at a point where both consumers' offer curves intersect. Figure 4(b) illustrates such a point. The upper-left point at which the two offer curves intersect is a Walrasian equilibrium allocation, and the price vector that ensures both players optimally choose the associated consumption bundles is a Walrasian equilibrium price vector. This example also illustrates that, relative to the Walrasian equilibrium allocation, there are no other feasible allocations that can make one of consumers better off without hurting the other consumer. The Walrasian equilibrium allocation is therefore Pareto optimal. Note that the offer curves also intersect at the endowment point, but the endowment point is not a Walrasian equilibrium allocation in this particular example—why?

The Edgeworth box is also useful for illustrating why there may be multiple Walrasian equilibria and why a Walrasian equilibrium might fail to exist. Figure 5(a) illustrates a situation in which there are multiple Walrasian equilibria. Here, the two consumers' offer curves intersect multiple times. Exercise 2 asks you to solve for the set of Walrasian equilibria in another example in which there are multiple equilibria.



Figures 5(a) and 5(b): multiple Walrasian equilibria and no Walrasian equilibria

Figure 5(b) illustrates a situation in which there are no Walrasian equilibria. In the example, consumer 1 is endowed with no units of commodity 1 and with all the units of commodity 2. Consumer 2 is endowed with all the units of commodity 1 and with no units of commodity 2. Consumer 2 cares only about her consumption of commodity 1, and consumer 2 cares about both her consumption of commodity 1 and her consumption of commodity 2. Moreover, consumer 1's marginal utility of consuming the first unit of commodity 1 is infinite, and her marginal utility of consuming commodity 2 is strictly positive. For any prices p with $p_1 > 0$, the market for commodity 1 cannot clear, since consumer 2 will always choose $x^*_{1,2}(p, p \cdot \omega_2) = \omega_{1,2}$, and consumer 1 will always choose $x^*_{1,1}(p, p \cdot \omega_1) > 0$, unless $p_2 = 0$. And if $p_2 = 0$, then $x^*_{2,1}(p, p \cdot \omega_1) = +\infty$, so the market for commodity 2 cannot clear. This example illustrates why things can go awry when assumption $(A4)$ is not satisfied. These examples tell us that the answers to the following two important questions is "no":

(a) is there always a Walrasian equilibrium? (b) if there is a Walrasian equilibrium, is it unique?

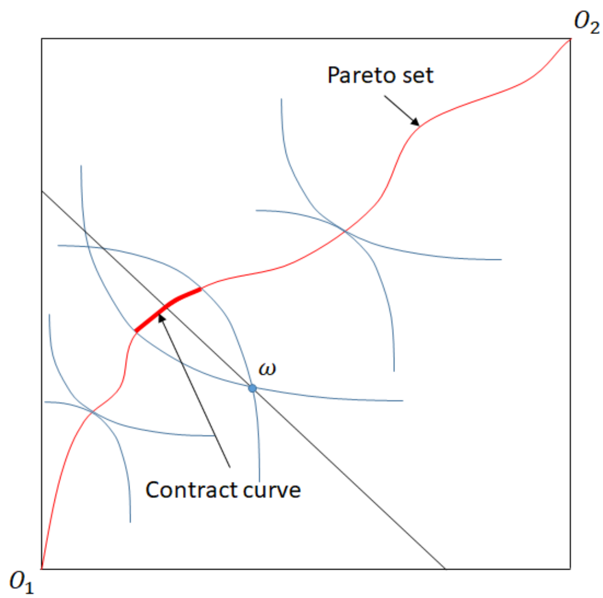


Figure 6: Pareto-optimal allocations and the contract curve

Finally, Figure 6 shows that we can use the Edgeworth box to illustrate the entire set of Pareto-optimal allocations. The **Pareto set** is the set of all feasible allocations for which making one consumer better off necessarily means making the other consumer worse off. It also illustrates the **contract curve**, which is the set of Pareto-optimal allocations that both players prefer to the endowment. If the two consumers were to negotiate a deal, given their endowments as their outside options, they would likely reach a point on the contract curve. Walrasian equilibrium allocations are typically a small subset of the contract curve, and in particular, they lie on the Pareto set. This result is known as the first welfare theorem, and we will now establish this result.

Exercise 1 (Adapted from MWG 15.B.2). Consider an Edgeworth box economy in which the consumers have Cobb-Douglas utility functions $u_1(x_{1,1}, x_{2,1}) = x_{1,1}^{\alpha} x_{2,1}^{1-\alpha}$ and $u_2(x_{1,2}, x_{2,2}) = x_{1,2}^{\beta} x_{2,2}^{1-\beta}$, where $\alpha, \beta \in (0,1)$. Consumer i's endowments are $(\omega_{1,i}, \omega_{2,i}) >> 0$

for $i = 1, 2$. Solve for the Walrasian equilibrium price ratio and allocation. How do these change as you increase $\omega_{1,1}$? Note: Feel free to avoid writing expressions out as much as possible. For example, if you solve for price, feel free to leave the solutions for demand in terms of the price variable instead of plugging in. For comparative statics, if you can find the sign without having to write it out, that's fine.

Exercise 2 (Adapted from MWG 15.B.6). Compute the Walrasian equilibria for the following Edgeworth box economy (there is more than one Walrasian equilibrium):

$$\begin{aligned} u_1(x_{1,1}, x_{2,1}) &= \left(x_{1,1}^{-2} + \left(\frac{12}{37}\right)^3 x_{2,1}^{-2}\right)^{-1/2}, & \omega_1 = (1, 0), \\ u_2(x_{1,2}, x_{2,2}) &= \left(\left(\frac{12}{37}\right)^3 x_{1,2}^{-2} + x_{2,2}^{-2}\right)^{-1/2}, & \omega_2 = (0, 1). \end{aligned}$$

Exercise 3 (Adapted from MWG 15.B.9). Suppose that in a pure exchange economy, we have two consumers, Alphanse and Betatrix, and two commodities, Perrier and Brie. Alphanse and Betatrix have the utility functions:

$$u_\alpha(x_{p,\alpha}, x_{b,\alpha}) = \min\{x_{p,\alpha}, x_{b,\alpha}\} \text{ and } u_\beta(x_{p,\beta}, x_{b,\beta}) = \min\left\{x_{p,\beta}, (x_{b,\beta})^{1/2}\right\},$$

(where $x_{p,\alpha}$ is Alphanse's consumption of Perrier, and so on). Alphanse starts with an endowment of 30 units of Perrier (and none of Brie); Betatrix starts with 20 units of Brie (and none of Perrier). Neither can consume negative amounts of a commodity. If the two consumers behave as price takers, what is the equilibrium? [Hint: consider the market-clearing condition in the cases when both prices are positive, when only the price of Perrier is positive, and when only the price of Brie is positive.]

Exercise 4. Consider an exchange economy with two consumers. The utility functions and endowments are given by

$$\begin{aligned} u_1(x_{1,1}, x_{2,1}) &= x_{1,1} - \frac{x_{2,1}^{-3}}{3}, & \omega_1 = (K, r) \\ u_2(x_{1,2}, x_{2,2}) &= x_{2,2} - \frac{x_{1,2}^{-3}}{3}, & \omega_2 = (r, K). \end{aligned}$$

Assume that K is sufficiently large so that each consumer can achieve an interior solution to her optimal consumption problem. Note that $p^* = (1, 1)$ is an equilibrium price vector.

(a) For what values of r will there be multiple Walrasian equilibria in this economy? [Hint: first solve for $q = p_y/p_x$ by showing that $rt^4 - t^3 + t - r = 0$, where $t = q^{1/4}$. Note this expression factors as $(t+1)(t-1)(rt^2 - t + r) = 0$.]

(b) For what value of r will $p^* = (1, 3)$ be an equilibrium price vector?

(c) [Optional: algebra intensive] Assume that $K = 10$ and that r takes the value identified in part (b). Find all equilibrium prices and allocations.

(d) [Optional: algebra intensive] Rank the outcomes identified in part (d) in terms of most preferred to least preferred for each consumer.

3 First Welfare Theorem

At the Walrasian equilibria in the examples we just saw, there are no feasible allocations that make both players better off: the Walrasian equilibrium allocation was Pareto optimal. This, it turns out, is a general result and perhaps one of the most important results of GE. This result is known as the first welfare theorem. Before stating and proving the first welfare theorem, we will first establish an intermediate result known as Walras's Law, which is a direct implication of consumer optimization when consumers' preferences are monotonic (or more generally, satisfy local non-satiation).

Lemma 1 (Walras's Law). Given an economy $\mathcal{E}$ and prices p, if $(A2)$ holds, then $p \cdot \left(\sum_{i\in\mathcal{I}} x_i \left(p, p \cdot \omega_i\right)\right) = p \cdot \left(\sum_{i\in\mathcal{I}} \omega_i\right)$.

Proof of Lemma 1. Since $(A2)$ holds, each consumer will optimally choose to exhaust her budget: $p \cdot x_i \left(p, p \cdot \omega_i\right) = p \cdot \omega_i$ for all $i \in \mathcal{I}$. Summing this condition over consumers gives us the expression in the Lemma.■

Note that Walras's Law holds for any set of allocations that are consumer-optimal—the result does not require that the allocation $\left(x_i \left(p, p \cdot \omega_i\right)\right)_{i\in\mathcal{I}}$ is a Walrasian equilibrium allocation.

Exercise 5 (Adapted from MWG 15.B.1). Consider an Edgeworth box economy in which the two consumers' preferences satisfy local nonsatiation. Let $x_{l,i}\left(p, p \cdot \omega_i\right)$ be consumer i's demand for commodity l at prices $p = (p_1, p_2)$.

(a) Show that $p_1 \sum_{i\in\mathcal{I}} \left(x_{1,l} - \omega_{1,l}\right) + p_2 \sum_{i\in\mathcal{I}} \left(x_{2,l} - \omega_{2,l}\right) = 0$ for all prices $p \neq 0$.

(b) Argue that if the market for commodity 1 clears at prices $p^* >> 0$, then so does the market for commodity 2; hence p^* is a Walrasian equilibrium price vector.

We can now prove a version of the first welfare theorem.

Theorem 1 (First Welfare Theorem). Suppose $\left(p^*, (x_i^*)_{i\in\mathcal{I}}\right)$ is a Walrasian equilibrium for the economy $\mathcal{E}$. Then if $(A2)$ holds, the allocation $(x_i^*)_{i\in\mathcal{I}}$ is Pareto optimal.

Proof of Theorem 1. In order to get a contradiction, suppose the Walrasian equilibrium allocation $(x_i^*)_{i\in\mathcal{I}}$ is not Pareto optimal. Then there is some other feasible allocation $(\hat{x}_i)_{i\in\mathcal{I}}$ for which $u_i(\hat{x}_i) \geq u_i(x_i^*)$ for all $i \in \mathcal{I}$ and $u_{i'}(\hat{x}_{i'}) > u_{i'}(x_{i'}^*)$ for some i'. Since $(x_i^*)_{i\in\mathcal{I}}$ is a Walrasian equilibrium allocation, and consumers' preferences satisfy $(A2)$, by revealed preference, it has to be the case that $p^* \cdot \hat{x}_i \geq p^* \cdot x_i^*$ for all $i \in \mathcal{I}$ and $p^* \cdot \hat{x}_{i'} > p^* \cdot x_{i'}^*$. Summing up over these conditions,

$$p^* \cdot \left(\sum_{i\in\mathcal{I}} \hat{x}_i\right) > p^* \cdot \left(\sum_{i\in\mathcal{I}} x_i^*\right) = p^* \cdot \left(\sum_{i\in I} \omega_i\right),$$

where the equality holds by Lemma 1. Since equilibrium prices p^* are nonnegative (why are they nonnegative?), this inequality implies that there is some commodity l such that $\sum_{i\in\mathcal{I}} \hat{x}_{i,l} > \sum_{i\in\mathcal{I}} \omega_{i,l}$, and therefore $(\hat{x}_i)_{i\in\mathcal{I}}$ is not a feasible allocation.■

The first welfare theorem is a remarkable result because (a) its conclusion is both intellectually important and powerful, (b) its explicit assumptions are quite weak, and (c) it has a simple proof, in the sense that it involves only a couple steps, and each step is completely transparent. Let me comment a bit more on each of these three points.

The first welfare theorem provides a formal statement of a version of Adam Smith's argument that the "invisible hand" of decentralized markets leads selfish consumers to make decisions that lead to socially efficient outcomes. Despite there being no explicit coordination among consumers, the resulting equilibrium allocation is Pareto optimal.

Second, the only explicit assumption we made in order to prove the first welfare theorem was that consumers have monotonic preferences—and even this assumption can be relaxed, as exercise 6 below asks you to show. But in the background, there are several strong and important assumptions. First, we assumed that all consumers face the same prices as each other for all commodities. Second, we assumed that all consumers are price takers—

they take prices as given and understand that their consumption decisions do not affect these prices. Third, there are markets for each commodity, and all consumers can freely participate in each market. Fourth, we assumed that each consumer cares only about her own consumption and not about the consumption of anyone else in the economy—we have therefore ruled out externalities. Finally, we assumed that there are a finite number of commodities and consumers. Exercise 6 asks you to show that when there are an infinite number of commodities and consumers, Walrasian equilibrium allocations need not be Pareto optimal.

Exercise 6 (Adapted from MWG 16.C.3). In this exercise, you are asked to establish the first welfare theorem under a set of assumptions compatible with satiation. First, we will define the appropriate notion of equilibrium. Given an economy $\mathcal{E}$, an allocation $(x_i^*)_{i\in\mathcal{I}}$ and a price vector $p = (p_1, \ldots, p_L)$ constitutes a **price equilibrium with transfers** if there is an assignment of wealth levels $(w_1, \ldots, w_I)$ with $\sum_{i\in\mathcal{I}} w_i = p \cdot \left(\sum_{i\in\mathcal{I}} \omega_i\right)$ such that: (i) consumers optimize: $x_i^*(p, w_i) = x_i^*$ and (ii) markets clear: $\sum_{i\in\mathcal{I}} x_i^* = \sum_{i\in\mathcal{I}} \omega_i$. Suppose that every $\mathcal{X}_i$ is nonempty and convex and that every u_i is strictly convex. Prove the following:

(a) For every consumer i, there is at most one consumption bundle at which she is locally satiated. Such a bundle, if it exists, uniquely maximizes u_i on $\mathcal{X}_i$.

(b) Any price equilibrium with transfers is a Pareto optimum.

Exercise 7. This exercise illustrates that the importance of the assumption that there are a finite number of commodities for the first welfare theorem. Consider an economy in which there is one physical good, available at infinitely many dates: $t = 1, 2, \ldots$, so there are effectively an infinite number of commodities: the physical good at date 1, the physical good at date 2, and so on. One consumer (or "generation") is born at each date $t = 1, 2, \ldots$, and lives and consumes at dates t and $t+1$ ("young" and "old"). We will refer to the consumer born in date t as consumer t. There is also one old consumer alive at date $t = 1$ (call her consumer 0). Each consumer is endowed with one unit of the good when she is young, and no storage is possible. Consumption in each period is non-negative, and each consumer t's preferences over consumption is given by $u_t(x_{t,t}, x_{t+1,t}) = u(x_{t,t}) + u(x_{t+1,t})$, where u is smooth, increasing, and strictly concave, with $u'(0) < \infty$.

(a) Show that there is a Walrasian equilibrium in which each consumer consumes her endowment and gets utility $u(1) + u(0)$.

(b) Show that the above Walrasian equilibrium is unique.

(c) Show that the above Walrasian equilibrium allocation is not Pareto optimal. In other words, construct a feasible allocation that is strictly better for each consumer.

I want to conclude this section with a couple comments on the simplicity of the proof of the first welfare theorem. First, the theorem itself presents a partial characterization of equilibrium allocations. To prove the statement, we did not need to solve explicitly for a Walrasian equilibrium and show that it is Pareto optimal. Instead, we described properties that all Walrasian equilibrium allocations must satisfy. Second, the statement itself is a conditional statement. It is a statement of the form "if $(x_i^*)_{i\in\mathcal{I}}$ is a Walrasian equilibrium, then $(x_i^*)_{i\in\mathcal{I}}$ is Pareto optimal." This conditional statement dodges the question of whether there is in fact a Walrasian equilibrium—we showed above that there does not always exist a Walrasian equilibrium, and we will spend some time next week providing conditions under which a Walrasian equilibrium in fact exists. Finally, the proof is a proof by contradiction, and it effectively takes the form of "if this Walrasian equilibrium allocation was not Pareto-optimal, then stuff doesn't add up." While elegant, the proof itself provides little insight into *why* the first welfare theorem holds. We will spend a little more time discussing the "why" in week 3.

4 Second Welfare Theorem

The first welfare theorem establishes that Walrasian equilibrium allocations are Pareto optimal. The second welfare theorem in some sense establishes a converse. It says that, under some assumptions, any Pareto optimal allocation can be "decentralized" as a Walrasian equilibrium allocation, given the correct prices and endowments.

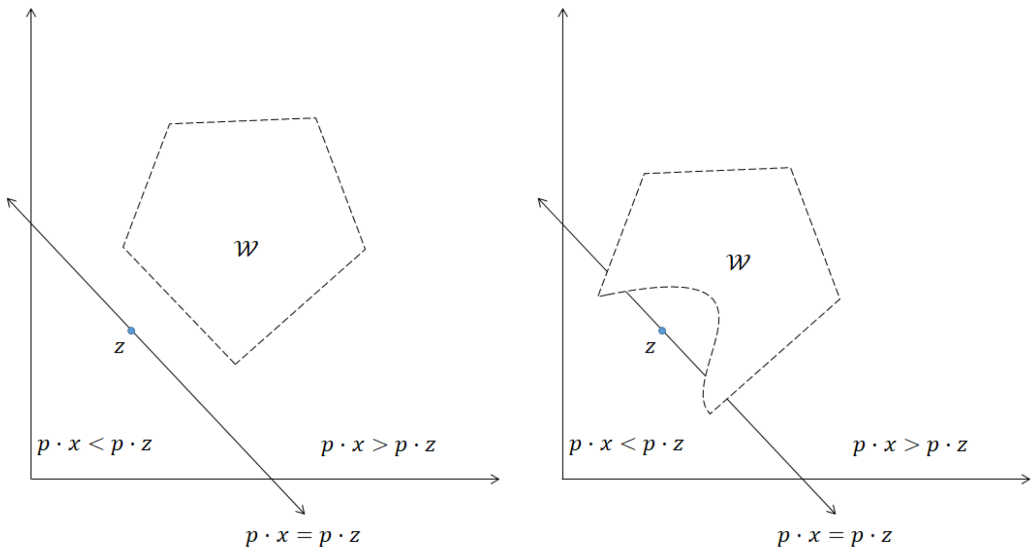
Theorem 2 (Second Welfare Theorem). Let $\mathcal{E}$ be an economy that satisfies $(A1)-(A4)$ and $\mathcal{X}_i = \mathbb{R}_+^L$. If $(\omega_i)_{i\in\mathcal{I}}$ is Pareto optimal, then there exists a price vector $p \in \mathbb{R}_+^L$ such that $\left(p, (\omega_i)_{i\in\mathcal{I}}\right)$ is a Walrasian equilibrium for $\mathcal{E}$.

Before proving the second welfare theorem, we will state a version of an important theorem in convex analysis, which is used in the key step of the proof of the second welfare theorem.

Lemma 2 (Separating Hyperplane Theorem). If $\mathcal{W} \subseteq \mathbb{R}^n$ is an open convex set, and $z \notin \mathcal{W}$ is a point not in $\mathcal{W}$, then there exists a vector $p \neq 0$ and such that $p \cdot x \geq p \cdot z$ for all $x \in cl(\mathcal{W})$.

Figure 7(a) illustrates this version of the separating hyperplane theorem in two dimensions. The set $\mathcal{W}$ is open and convex, and $z \notin \mathcal{W}$. The point z is on a line (which is a hyperplane in a two-dimensional space) characterized by the equation $p \cdot x = p \cdot z$ (i.e., p is the normal vector to the line). All the points to the upper right of that line satisfy $p \cdot x > p \cdot z$, and all the points to the lower left of that line satisfy $p \cdot x < p \cdot z$. And in particular, $\mathcal{W}$ is fully to the upper right of this line. In the case illustrated in Figure 7(a), there are of course many other separating hyperplanes satisfying $p \cdot x \geq p \cdot z$ for all $x \in \mathcal{W}$ corresponding to differently sloped lines going through z but not intersecting $\mathcal{W}$. Figure 7(b) shows why the assumption that $\mathcal{W}$ is a convex set is important for this result. If $z \notin \mathcal{W}$, but $z \in conv(\mathcal{W})$,[3] then there is no vector $p \neq 0$ for which $p \cdot x \geq p \cdot z$ for all $x \in \mathcal{W}$. Exercise 8 asks you to prove a stronger version of the separating hyperplane theorem, which shows that any two disjoint convex sets can be separated by a hyperplane.

[3]The set $conv(\mathcal{W})$ is defined to be the smallest convex set containing $\mathcal{W}$. In two dimensions, you can visualize $conv(\mathcal{W})$ by taking $\mathcal{W}$ and putting a rubber band around it.



Figures 7(a) and 7(b): separating hyperplane theorem and convexity.

The idea of the second welfare theorem is to show that, if the endowment $(\omega_i)_{i\in\mathcal{I}}$ is Pareto optimal, we can always find a price vector that separates the set of allocations preferred by all consumers in the economy from $(\omega_i)_{i\in\mathcal{I}}$ and therefore show that $\left(p, (\omega_i)_{i\in\mathcal{I}}\right)$ is a Walrasian equilibrium.

Proof of Theorem 2. By the statement of the theorem, $(\omega_i)_{i\in\mathcal{I}}$ is Pareto optimal. Let us define the set of aggregate consumption bundles that can be allocated in such a way among consumers to make them all strictly better off than under $(\omega_i)_{i\in\mathcal{I}}$. To do so, define the set of consumption bundles that consumer i prefers to ω_i:

$$\mathcal{A}_i = \left\{a \in \mathbb{R}^L : a + \omega_i \geq 0 \text{ and } u_i\left(a + \omega_i\right) > u_i\left(\omega_i\right)\right\}.$$

Since u_i is concave, the set $\mathcal{A}_i$ is convex. The Minkowski sum of the sets $\mathcal{A}_i$ is therefore also a convex set.[4] That is, if we define

$$\mathcal{A} = \sum_{i \in \mathcal{I}} \mathcal{A}_i = \left\{ a \in \mathbb{R}^L : \exists a_1 \in \mathcal{A}_1, \ldots, \exists a_I \in \mathcal{A}_I \text{ with } a = \sum_{i \in \mathcal{I}} a_i \right\},$$

then $\mathcal{A}$ is a convex set. The set $\mathcal{A}$ does not contain the 0 vector because $(\omega_i)_{i \in \mathcal{I}}$ is Pareto optimal. To see why this is the case, note that if $0 \in \mathcal{A}$, then there would exist $(a_i)_{i \in \mathcal{I}}$ with $\sum_{i \in \mathcal{I}} a_i = 0$ and $u_i(a_i + \omega_i) > u_i(\omega_i)$ for all i. That is, we could essentially just reallocate the endowment $(\omega_i)_{i \in \mathcal{I}}$ among the I consumers and make them all strictly better off, but that would contradict the assumption that $(\omega_i)_{i \in \mathcal{I}}$ is Pareto optimal.

Next, by Lemma 2, there is some price vector $p^* \neq 0$ such that $p^* \cdot a \geq 0$ for all $a \in cl(\mathcal{A})$. Moreover, each of the prices $p_l^* \geq 0$. To see why, suppose $p_l^* < 0$ for some l. Take some a for which a_l is arbitrarily large and all other $a_{l'}$ are arbitrarily small but positive. By the monotonicity of consumer preferences, $a \in \mathcal{A}$, but if a is chosen this way, then $p \cdot a < 0$. We therefore have that $p^* > 0$ (i.e., $p_l^* \geq 0$ for all $l \in \mathcal{L}$ with at least one inequality strict).

We will now show that $\left(p^*, (\omega_i)_{i \in \mathcal{I}}\right)$ is a Walrasian equilibrium. To do so, we need to show that at p^*, consumers optimally consume their endowments and that markets clear. The second condition is immediate. It remains to show that at this p^*, consumers optimally consume their endowments. To do so, suppose there is some $\hat{x}_i \in \mathbb{R}_+^L$ for which $u_i(\hat{x}_i) > u_i(\omega_i)$. We will show that this $\hat{x}_i \notin \mathcal{B}_i(p^*)$. By the definition of $\mathcal{A}$, the allocation $(x_i)_{i \in \mathcal{I}} - (\omega_i)_{i \in \mathcal{I}}$ with $x_i = \hat{x}_i$ and $x_j = \omega_j$ for all $j \neq i$, is in $cl(\mathcal{A})$. By the definition of p^*, we necessarily have that $p^* \cdot (\hat{x}_i - \omega_i) + p^* \cdot \sum_{j \neq i} (\omega_j - \omega_j) \geq 0$, which implies that $p^* \cdot \hat{x}_i \geq p^* \cdot \omega_i > 0$, where this last inequality holds because of Assumption $(A4)$ that all consumers have positive endowments of all commodities.

We are not yet done, because we have to show that this last inequality is strict. This is where continuity of preferences (Assumption $(A1)$) comes into the picture. Since $u_i(\hat{x}_i) >$

[4] The Minkowski sum of two sets $\mathcal{A}$ and $\mathcal{B}$ is just the set of vectors x that can be written as the sum of vectors $x = a + b$ for which $a \in \mathcal{A}$ and $b \in \mathcal{B}$. The closest visual analog to thinking about the Minkowski sum of sets in two dimensions is the way the clone stamp tool in Photoshop works if you are familiar with it.

$u_i(\omega_i)$, this implies that for λ just less than 1, $u_i(\lambda \hat{x}_i) > u_i(\omega_i)$, which in turn implies that $\lambda p^* \cdot \hat{x}_i \geq p^* \cdot \omega_i > 0$. This cannot be the case if $p^* \cdot \hat{x}_i = p^* \cdot \omega_i$, so we must therefore have that $p^* \cdot \hat{x}_i > p^* \cdot \omega_i$ and hence $\hat{x}_i \notin \mathcal{B}_i(p^*)$—that is, any allocation preferred by consumer i to her endowment is unaffordable, and hence her optimal consumption bundle is her endowment.■

The second welfare theorem does not show that every Pareto optimal allocation is a Walrasian equilibrium given a *particular* endowment. Instead, it says that if we were to start from a particular endowment $(\omega_i)_{i \in \mathcal{I}}$, and an allocation $(x_i)_{i \in \mathcal{I}}$ is Pareto optimal, then we could reallocate consumers' endowments in such a way that $(x_i)_{i \in \mathcal{I}}$ is a Walrasian equilibrium allocation. The version of the theorem that we just proved carries out this exercise using a particularly stark reallocation of endowments (i.e., it just sets $(\omega_i)_{i \in \mathcal{I}} = (x_i)_{i \in \mathcal{I}}$). There are versions of the theorem that involve carrying out lump-sum transfers of wealth rather than directly moving around endowments. As you might expect, decentralizing a particular Pareto-optimal allocation in practice potentially requires large-scale redistribution of wealth. I view the result more as establishing an equivalence between Walrasian equilibria and Pareto-optimal allocations rather than as a practical guide for figuring out how to achieve a particular distribution of consumption in society.

It is worth a reminder that convexity of consumers' preferences was critical in establishing the result that $\mathcal{A}$ was a convex set, which in turn is required for using the separating hyperplane theorem. Figure 8 shows an example where the conclusion of the second welfare theorem fails if consumers' preferences are not convex.

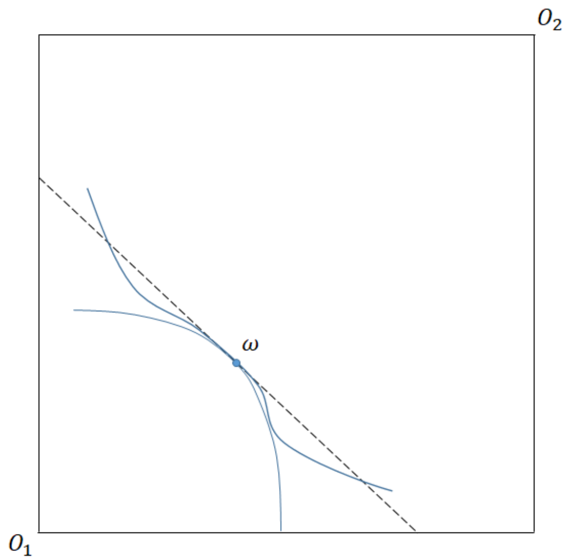


Figure 8: Non-convex preferences

In this figure, the endowment is a Pareto-optimal allocation, since consumer 1's and consumer 2's better-than sets are separated. But there are no prices that can make it optimal for consumer 1 to consume ω_1.

Nevertheless, a version of the second welfare theorem continues to hold when consumers do not have convex preferences if you replicate the economy a large number of times. Think of the 2-consumer economy as being a metaphor for a large economy with two *types* of consumers: type-1 consumers have preferences u_1 and endowments ω_1, and type-2 consumers have preferences u_2 and endowments ω_2. If we replicate the economy a large number of times, so that there are N type-1 consumers and N type-2 consumers, where N is large, then we can support ω as a Walrasian equilibrium allocation, at least on average. This result follows from an application of the Shapley-Folkman lemma, which roughly says that the Minkowski average of sets converges to the convex hull of that set. You don't need to know the math behind this result, but it is a useful result to be aware of. Figure 9 illustrates a replication economy for the economy described in Figure 8. It shows that there may be a p for which there is an $x_1 \in x_1(p, p \cdot \omega_1)$ and an $x_1' \in x_1(p, p \cdot \omega_1)$, so that if we allocate a fraction λ of type-1 consumers to consume x_1 and a fraction $1-\lambda$ of type-1 consumers to consume x_1', on average they are consuming ω_1: $\lambda x_1 + (1-\lambda) x_1' = \omega_1$.

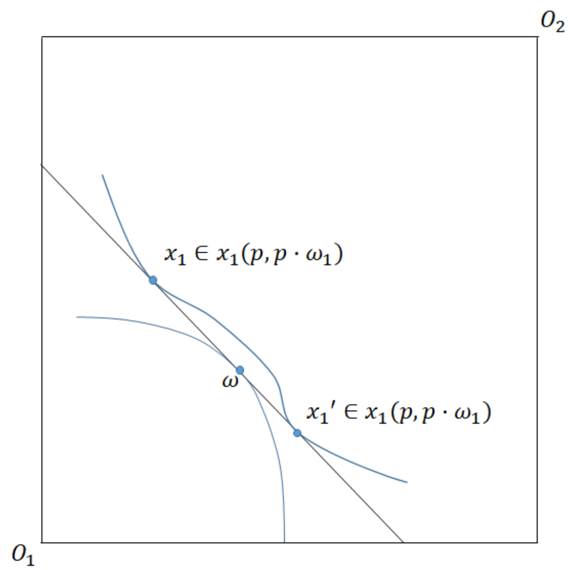


Figure 9: Replication economy

This figure illustrates the idea that large numbers "convexifies" the economy. There is a recurring theme throughout general equilibrium theory that many of the pathologies that arise seem to "go away" in sufficiently large economies. Nonconvexities seem esoteric, since we usually think of consumers' preferences as having diminishing marginal utility and preferences for variety. Nonconvexities become especially relevant when we think of firms, though. When there are fixed costs, for example, the firm analogue of consumers' "better-than" sets are not convex, since the set of production levels better than "not even breaking even" can include both "shut down" and "produce, but at a much larger scale."

Finally, we made use of Assumption $(A4)$ in a somewhat opaque way in the proof. What Assumption $(A4)$ rules out is cases like the one illustrated in Figure 5(b) in which there were no Walrasian equilibria. The failure of equilibrium existence illustrated in Figure 5(b) arises because of a sort of "division by zero" problem: supporting the endowment as an equilibrium allocation would have required consumer 2 to buy only a finite amount of a commodity with a zero price when she has zero wealth.

Exercise 8. This question is intended to guide you through a proof of the separating hyperplane theorem. This is more of an exercise in math than in economics, so feel free to skip to the next step if you get stuck.

(a) Prove that if $y \in \mathbb{R}^N$ and $\mathcal{C} \subseteq \mathbb{R}^N$ is closed, then there exists a point $z \in \mathcal{C}$ such that $||z - y|| \leq ||x - y||$ for all $x \in \mathcal{C}$. That is, there exists a point in $\mathcal{C}$ that is closest to y. (You may assume that $||\cdot||$ is the Euclidean norm.) Hint: use the Weierstrass extreme value theorem—if f is a real-valued and continuous function on domain $\mathcal{S}$, and $\mathcal{S}$ is compact and non-empty, then there exists x such that $f(x) \geq f(y)$ for all $y \in \mathcal{S}$.

(b) Suppose further that $\mathcal{C} \subseteq \mathbb{R}^N$ is convex, and note from above that if $y \notin \mathcal{C}$, then there exists $z \in \mathcal{C}$ that is closest to y. Let $x \in \mathcal{C}$ with $x \neq z$.

(i) Show that $(y - z) \cdot z \geq (y - z) \cdot x$. Hint: consider $||y - (z + t(x - z))||$ for $t \in [0, 1]$, the distance between y and a convex combination of x and z.

(ii) Use the above result to show that for all $x \in \mathcal{C}$, $(y - z) \cdot y > (y - z) \cdot x$.

(iii) Explain how this is a special case of the separating hyperplane theorem, which states that for any disjoint convex sets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^N$, there exists nonzero $p \in \mathbb{R}^N$ such that $p \cdot u \geq p \cdot v$ for any $u \in \mathcal{A}$ and $v \in \mathcal{B}$.

(iv) Use the result of (ii) to deduce the separating hyperplane theorem. Hint: consider $y = 0$ and $\mathcal{C} = \mathcal{A} - \mathcal{B} = \{u - v : u \in \mathcal{A}, v \in \mathcal{B}\}$.

5 Characterizing Pareto-Optimal Allocations

The welfare theorems provide a tight connection between the set of Pareto optimal allocations and the set of Walrasian equilibrium allocations. This section will provide a short note on how to find Pareto optimal allocations in particularly well-behaved environments. Define the **utility possibility set**

$$\mathcal{U} = \left\{(u_1, \ldots, u_I) \in \mathbb{R}^I : \text{there is a feasible allocation } (x_i)_{i \in \mathcal{I}} \text{ with } u_i(x_i) \geq u_i \text{ for all } i\right\}.$$

If the sets $\mathcal{X}_i$ are convex sets and consumers' preferences are concave, then $\mathcal{U}$ is a convex set. When this is the case, the problem of finding Pareto-optimal allocations can be reduced to the problem of solving **Pareto problems** of the form

$$\max_{u \in U} \lambda \cdot u$$

for some non-zero vector of **Pareto weights** $\lambda \geq 0$. The objective function of this problem is sometimes called a linear Bergson-Samuelson social welfare function. We will say that u^* is a **Pareto-optimal utility vector** if there is a Pareto-optimal allocation $(x_i)_{i \in \mathcal{I}}$ for which $u_i(x_i) = u_i^*$ for all $i \in \mathcal{I}$. The next theorem establishes the result.

Theorem 3. If u^* is a solution to the Pareto problem described above for some vector of Pareto weights $\lambda >> 0$, then u^* is a Pareto-optimal utility vector. Conversely, if the utility possibility set $\mathcal{U}$ is convex, then any Pareto-optimal utility vector u^* is a solution to the Pareto problem for some non-zero vector $\lambda \geq 0$.

Proof of Theorem 3. The first part is immediate: if u^* is not Pareto optimal, then any Pareto-dominating utility vector would give a higher value in the Pareto problem for any Pareto weight vector $\lambda >> 0$.

The second part of the theorem makes use of the *supporting hyperplane theorem*, which says that a convex set can be separated from any point outside its interior (see Section M.G of the mathematical appendix of MWG). If u^* is a Pareto-optimal utility vector, then it lies on the boundary of $\mathcal{U}$, so by the supporting hyperplane theorem, there exists $\lambda \neq 0$ such that $\lambda \cdot u^* \geq \lambda \cdot u$ for all $u \in \mathcal{U}$. Further, the Pareto weights satisfy $\lambda \geq 0$, since if $\lambda_i < 0$ for some i, then $\lambda \cdot u^* < \lambda \cdot \tilde{u}$, where for some $K > 0$, $\tilde{u} = \left(u_1^*, \ldots, u_{i-1}^*, u_i^* - K, u_{i+1}^*, \ldots, u_I^*\right) \in \mathcal{U}$. This contradicts the claim that $\lambda \cdot u^* \geq \lambda \cdot u$ for all $u \in \mathcal{U}$, so it must be the case that $\lambda \geq 0$.■

The theorem shows that when the utility possibility set is a convex set, the problem of finding Pareto-optimal allocations boils down to solving a class of Pareto problems. If we further assume that consumers' utility functions are differentiable, then Pareto-optimal allocations can be characterized by taking first-order conditions. For example, suppose utility functions are differentiable with $\nabla u_i(x_i) >> 0$ for all x_i, and we have an interior solution, we can find Pareto-optimal allocations by solving the problem:

$$\max_{(x_i)_{i\in\mathcal{I}}} \sum_{i\in\mathcal{I}} \lambda_i u_i(x_i)$$

subject to feasibility for each commodity:

$$\sum_{i\in\mathcal{I}} x_{l,i} \leq \sum_{i\in\mathcal{I}} \omega_{l,i} \text{ for all } l \in \mathcal{L}.$$

Then one can use the Kuhn-Tucker theorem to verify that any Pareto-optimal allocation $(x_i)_{i\in\mathcal{I}}$ with $x_i >> 0$ for all $i \in \mathcal{I}$ must satisfy

$$\frac{\partial u_i/\partial x_{l,i}}{\partial u_i/\partial x_{l',i}} = \frac{\partial u_{i'}/\partial x_{l,i'}}{\partial u_{i'}/\partial x_{l',i'}} = \frac{\mu_l}{\mu_{l'}} \text{ for all } i, i', l, l'$$

for some $\mu_l, \mu_{l'} > 0$. This condition says that the marginal rate of substitution between any two commodities must be equalized across consumers in any Pareto-optimal allocation. If this condition failed, there would be a Pareto-improving exchange of commodities l and l' between consumers i and i'. The values μ_l corresponds to the Lagrange multiplier on the commodity-l feasibility constraint $\sum_{i\in\mathcal{I}} x_{l,i} = \sum_{i\in\mathcal{I}} \omega_{l,i}$.

As an illustration of the second welfare theorem, given a Pareto-optimal allocation $(x_i)_{i\in\mathcal{I}}$ that satisfies the optimality conditions above, if you set $p_l = \mu_l$ for all $l \in \mathcal{L}$, then $\left(p, (x_i)_{i\in\mathcal{I}}\right)$ is a Walrasian equilibrium of the economy $\mathcal{E} = \left((u_i)_{i\in\mathcal{I}}, (x_i)_{i\in\mathcal{I}}\right)$. This point is illustrated in Figure 4(b). In that figure, at the Walrasian equilibrium allocation, consumers' marginal rates of substitution across the two commodities were equalized. Moreover, these marginal rates of substitution were also equal to the price ratio that corresponded to the Walrasian equilibrium (and given that price ratio, moving the endowment along the boundary of consumers' budget sets does not change their ultimate consumption choices, so the same price vector would also be an equilibrium price vector if we just set consumers' endowments equal to their Walrasian equilibrium allocations).

Exercise 9 (Adapted from MWG 16.C.4). Suppose that for each consumer, there is a "happiness function" depending on her own consumption only, given by $u(x_i)$. Every consumer's utility depends positively on her own and everyone else's "happiness" according to the utility function

$$U_i(x_1, \ldots, x_l) = U_i(u_1(x_1), \ldots, u_l(x_l)).$$

Show that if $x = (x_1, \ldots, x_l)$ is Pareto optimal relative to the $U_i(\cdot)$'s, then $x = (x_1, \ldots, x_l)$ is also a Pareto optimum relative to the u_i's. Does this mean a community of altruists can use competitive markets to attain Pareto optima? Does your argument depend on the concavity of the u_i's or the U_i's?