Economics 2010b: Microeconomic Theory II

Michael Powell

General Equilibrium Theory

1 Introduction to General Equilibrium Theory[1]

In the first three weeks of this course, our goal is to develop a parsimonious model of the overall economy to study the interaction of individual consumers and firms in perfectly competitive *decentralized* markets. The resulting framework has provided the workhorse micro-foundations for much of modern macroeconomics, international trade, and financial economics. In the last three and a half weeks, we will begin to study *managed* transactions and in particular how individuals should design institutions such as contracts and property rights allocations to governance structures, to achieve desirable outcomes.

The main ideas of general equilibrium theory have a long history, going back to Adam Smith's (1776) evocative descriptions of how competition channels individual self-interest in the social interest and how a sense of "coherence among the vast numbers of individuals and seemingly separate decisions" (Arrow, 1972) can arise in the economy without explicit design. General equilibrium theory addresses how this aggregate "coherence" emerges from individual interactions and can potentially lead to socially desirable allocations of goods and services in the economy. The mechanism through which this coherence emerges is, of course, the price mechanism. Individuals facing the same, suitably determined, prices will end up making decisions that are well-coordinated at the economy-wide level.

What distinguishes general equilibrium theory from partial equilibrium theory, which you have studied in Economics 2010a, is the idea that if we want to develop a theory of the price system for the economy as a whole, we have to consider the equilibrium in all markets in the economy simultaneously. As you can imagine, thinking about all markets simultaneously can be a complicated endeavor, since markets are interdependent: the price of computer chips

[1] These notes borrow liberally from Levin's (2006) notes and Wolitzky's (2016) notes.

will affect the price of software, cars, appliances, and so on. General equilibrium theory was the most active research area in economic theory for a good part of the 20th century and is therefore a very rich topic. Our goal over the first three weeks of the course is to cover only the basics.

In many ways, the first part of this class will be structured the way an applied theory paper is structured. We will start by talking about the setup of a model: who are the players, what do they do, what do they know, what are their preferences, what is the solution concept we will be using? Then we will partially characterize its solution, focusing on its efficiency properties in particular. What do I mean by partially characterize? I mean that we will be describing some properties of equilibria that we can talk about without actually solving the full model. That should take us through the first week.

In the second week, we will do a little bit of heavier lifting and begin with a question that is often easy to overlook but is very important to answer: does an equilibrium exist? Existence proofs can sometimes seem like a bit of an esoteric detour, but a good existence proof is especially useful if it helps build tools to answer other important questions about the model. After we establish that an equilibrium exists, we will ask questions like: when is there a unique equilibrium? Are equilibria stable? What are the testable implications of equilibrium behavior?

The third week will focus on what I will blandly call "extensions," but are really the meat and potatoes of how general equilibrium theory gets used in practice. We will show how we can introduce firms, time, and uncertainty into the framework, and we will talk about how and when the main results we identified above apply in these settings as well. And finally, we will end by putting the model's solution concept, competitive equilibrium, on firmer microfoundations. A lot of economics is buried in the solution concept, and developing microfoundations for the solution concept is a useful way to flesh out some of the key insights.

2 Pure Exchange Economies

A general equilibrium model describes three basic activities that take place in the economy: production, exchange, and consumption. For the first two weeks, we will set production aside and focus on the minimal number of modeling ingredients necessary to give you a flavor of the powerful results of general equilibrium theory.

We begin with a description of the model we will be using. As always, the exposition of an economic model specifies a complete description of the economic environment (players, actions, and preferences) and the solution concept that will be used to derive prescriptions and predictions.

2.1 The Model

Formally, a pure exchange economy is an economy in which there are no production opportunities. There are I **consumers** $i \in \mathcal{I} = \{1, \ldots, I\}$ who buy, sell, and consume L **commodities**, $l \in \mathcal{L} = \{1, \ldots, L\}$. A **consumption bundle for consumer** i is a vector $x_i = (x_{1,i}, \ldots, x_{L,i})$, where $x_i \in \mathcal{X}_i$, which is consumer i's **consumption set** and just describes her feasible consumption bundles. We will assume throughout that $\mathcal{X}_i$ contains the 0 vector and is a convex set. Consumer i has an **endowment** of the L commodities, which is described by a vector $\omega_i = (\omega_{1,i}, \ldots, \omega_{L,i})$ and has preferences over consumption bundles, which we assume can be represented by a utility function $u_i : \mathcal{X}_i \to \mathbb{R}$. A **pure exchange economy** is therefore a set $\mathcal{E} = \left((u_i, \omega_i)_{i \in \mathcal{I}}\right)$, which fully describes the model's primitives: the set of players, their preferences, and their endowments.

Each consumer takes **prices** $p = (p_1, \ldots, p_L)$ as given and solve her **consumer maximization problem**:

$$\max_{x_i \in \mathcal{X}_i} u_i(x_i) \text{ s.t. } p \cdot x_i \leq p \cdot \omega_i,$$

where the right-hand side of the consumer's budget constraint is her wealth, as measured by the market value of her endowment at prices p. Consumer i's feasible actions are therefore

$x_i \in \mathcal{B}_i(p) \equiv \{x_i \in \mathcal{X}_i : p \cdot x_i \leq p \cdot \omega_i\}$, where we refer to $\mathcal{B}_i(p)$ as her **budget set at prices** p. Given prices p and endowment ω_i, we will refer to consumer i's optimal choices as her **Marshallian demand correspondence** and denote it by $x_i(p, p \cdot \omega_i)$. Simply put, all consumers do in this model is to choose their favorite consumption bundles in their budget sets.

We have now described the players and their actions, but no model description is complete without a solution concept. Here, the solution concept will be a Walrasian equilibrium, which will specify a set of prices and a consumption bundle for each consumer (we will refer to a collection of consumption bundles for each consumer as an **allocation**) that satisfy two properties: consumer optimization and market-clearing. Given prices p, each consumer optimally chooses her consumption bundle, and total demand for each commodity equals total supply.

Definition 1. *A* ***Walrasian equilibrium*** *for the pure exchange economy* $\mathcal{E}$ *is a vector* $\left(p^*, (x_i^*)_{i \in \mathcal{I}}\right)$ *that satisfies:*

1. *Consumer optimization: for all consumers* $i \in \mathcal{I}$,

$$x_i^* \in \operatorname{argmax}_{x_i \in \mathcal{B}_i(p^*)} u_i(x_i),$$

2. *Market-clearing: for all commodities* $l \in \mathcal{L}$,

$$\sum_{i \in \mathcal{I}} x_{l,i}^* = \sum_{i \in \mathcal{I}} \omega_{l,i}.$$

We have now fully specified the model, but before we start to go into more detail discussing the properties of Walrasian equilibria, there are a couple more important definitions to introduce. The first is the notion of a feasible allocation, which is just a collection of consumption bundles for each consumer for which the total amount consumed for each commodity does not exceed the total endowment of that commodity.

Definition 2. *An allocation* $(x_i)_{i\in\mathcal{I}} \in R_+^{L\cdot I}$ *is* ***feasible*** *if for all* $l \in \mathcal{L}$, $\sum_{i\in\mathcal{I}} x_{l,i} \leq \sum_{i\in\mathcal{I}} \omega_{l,i}$.

The next definition is going to describe how we will be thinking about optimality in the economy. For a lot of optimization problems you have seen in your other courses, the appropriate notion of optimality is straightforward. For example, if a consumer has a well-defined utility function, it is straightforward to think about what is optimal for her given her budget set. Once we start thinking about environments with more than one consumer, we would, in some sense, like to maximize multiple objective functions (i.e., each consumer's utility) simultaneously. In general, there are no allocations that simultaneously maximize the utility of all consumers—consumers' objectives are typically in conflict with one another's—so the appropriate notion of optimality is not as straightforward. The notion will we use is that of Pareto optimality, which means that all we are doing is ruling out allocations that are dominated by other feasible allocations.

Definition 3. *Given an economy* $\mathcal{E}$, *a feasible allocation* $(x_i)_{i\in\mathcal{I}}$ *is* ***Pareto optimal*** *(or* ***Pareto efficient****) if there is no other feasible allocation* $(\hat{x}_i)_{i\in\mathcal{I}}$ *such that* $u_i(\hat{x}_i) \geq u_i(x_i)$ *for all* $i \in \mathcal{I}$ *with strict inequality for some* $i \in \mathcal{I}$.

In words, all Pareto optimality rules out is allocations for which someone could be made better off without making anyone else worse off. This notion of optimality is therefore silent on issues of distribution, since it may be Pareto optimal for one consumer to consume everything in the economy and for everyone else to consume nothing.

2.2 Assumptions on Consumer Preferences and Endowments

Throughout the next few sections, we will invoke different sets of assumptions for different results. I will collect these assumptions here and will be explicit in referring to them when they are required for a result.

Assumption A1 (continuity): For all consumers $i \in \mathcal{I}$, u_i is continuous.

Assumption A2 (monotonicity): For all consumers $i \in \mathcal{I}$, u_i is increasing: $u_i(x_i') >$

$u_i(x_i)$ whenever $x'_{l,i} > x_{l,i}$ for all $l \in \mathcal{L}$.

Assumption A3 (concavity): For all consumers $i \in \mathcal{I}$, u_i is concave.

Assumption A4 (interior endowments): For all consumers $i \in \mathcal{I}$, $\omega_{l,i} > 0$ for all $l \in \mathcal{L}$.

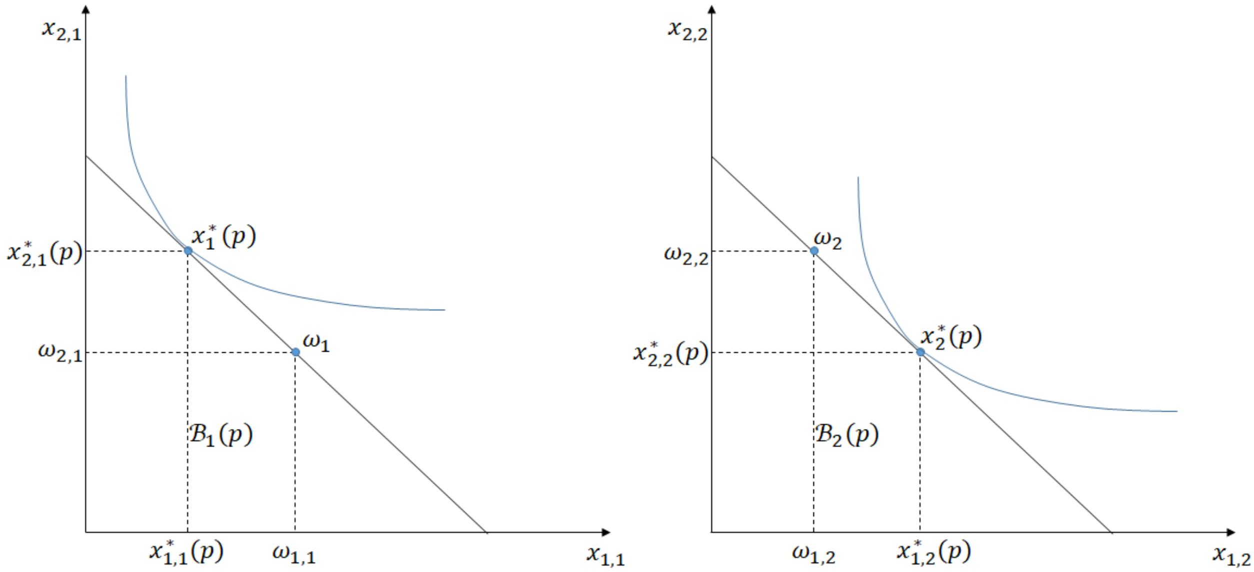
The first three assumptions should be familiar from Economics 2010a. The results we will be establishing in the upcoming sections will hold under weaker assumptions—for example, ($A2$) can typically be relaxed to local nonsatiation,[2] and ($A3$) can typically be relaxed to quasiconcavity. The last assumption is a strong assumption that will prove to be sufficient for ruling out some pathological cases in which a Walrasian equilibrium does not exist.

2.3 Graphical Examples

Many of the main ideas of general equilibrium theory can be understood in a two-consumer, two-commodity pure exchange economy. We can get most of the results across graphically in what is referred to as an Edgeworth box. Edgeworth boxes are informationally dense, so let me introduce the constituent elements separately.

Figure 1(a) depicts the relevant information for consumer 1. On the horizontal axis is her consumption of commodity 1 and on the vertical axis is her consumption of commodity 2. Her endowment is $\omega_1 = (\omega_{1,1}, \omega_{2,1})$. At prices $p = (p_1, p_2)$, she can afford to buy any consumption bundle in the set $\mathcal{B}_1(p)$. The slope of her budget line is $-p_1/p_2$. Given her preferences, which are represented by her indifference curve, and given prices p and endowment ω_1, she optimally chooses to consume $x_1^*(p) = \left(x_{1,1}^*(p), x_{2,1}^*(p)\right)$. In other words, at prices p, she would optimally like to sell $\omega_{1,1} - x_{1,1}^*(p)$ units of commodity 1 in exchange for $x_{2,1}^*(p) - \omega_{2,1}$ units of commodity 2. Figure 1(b) depicts the same information for consumer 2.

[2] We say that consumer i's preferences satisfy **local non-satiation** if for every $x_i \in \mathcal{X}_i$ and every $\varepsilon > 0$, there is an $x'_i \in \mathcal{X}_i$ such that $||x'_i - x_i|| \leq \varepsilon$ and $u_i(x'_i) > u_i(x_i)$.



Figures 1(a) and 1(b): consumer-optimization problems

The Edgeworth box represents both consumers' endowments and their optimal choices as a function of the prices p, so it will incorporate all the information in Figures 1(a) and 1(b). To build towards this goal, Figure 2(a) depicts all the **non-wasteful allocations** in the economy: allocations $(x_i)_{i\in\{1,2\}}$ for which $x_{l,1} + x_{l,2} = \omega_{l,1} + \omega_{l,2}$ for $l \in \{1, 2\}$. The bottom-left corner is the origin for consumer 1, and the upper-right corner is the origin for consumer 2. The length of the horizontal axis is equal to the total endowment of commodity 1, and the length of the vertical axis is equal to the total endowment of commodity 2. The horizontal axis, read from the left to the right, represents consumer 1's consumption of commodity 1, and read from the right to the left, represents consumer 2's consumption of commodity 1. The vertical axis, read from the bottom to the top, represents consumer 1's consumption of commodity 2, and read from the top to the bottom, represents consumer 2's consumption of commodity 2. The endowment ω is a point in the Edgeworth box, and it represents a non-wasteful allocation, since $\omega_{l,1} + \omega_{l,2} = \omega_{l,1} + \omega_{l,2}$ for $l \in \{1, 2\}$. The allocation x also represents a non-wasteful allocation, since $x_{l,1} + x_{l,2} = \omega_{l,1} + \omega_{l,2}$ for $l \in \{1, 2\}$.

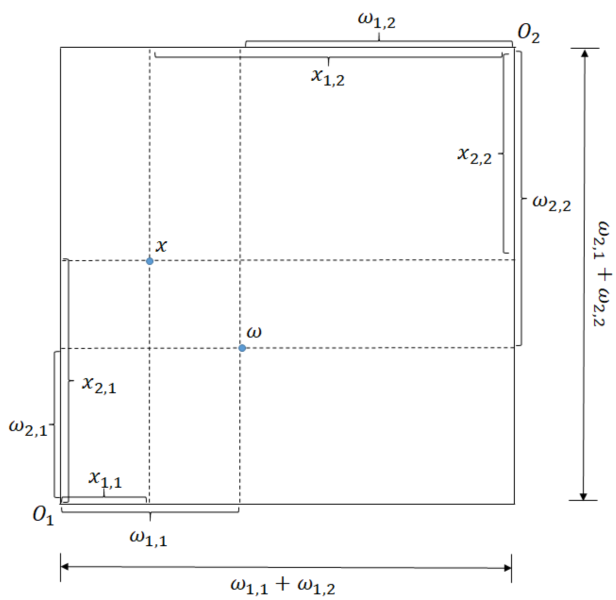


Figure 2(a): non-wasteful allocations

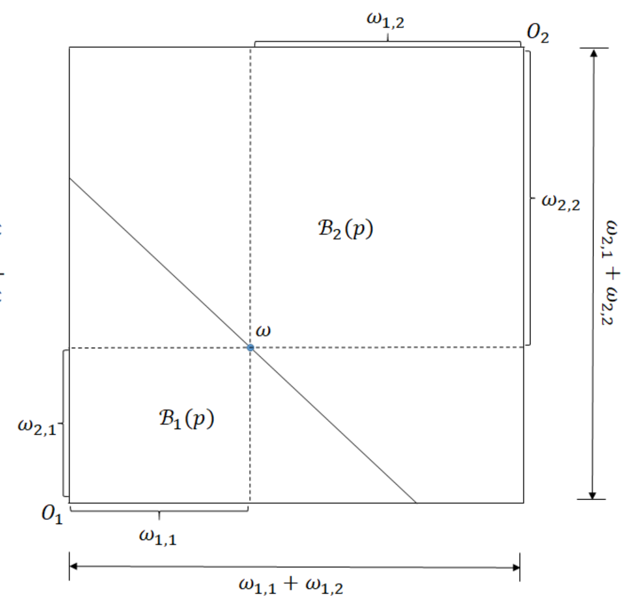
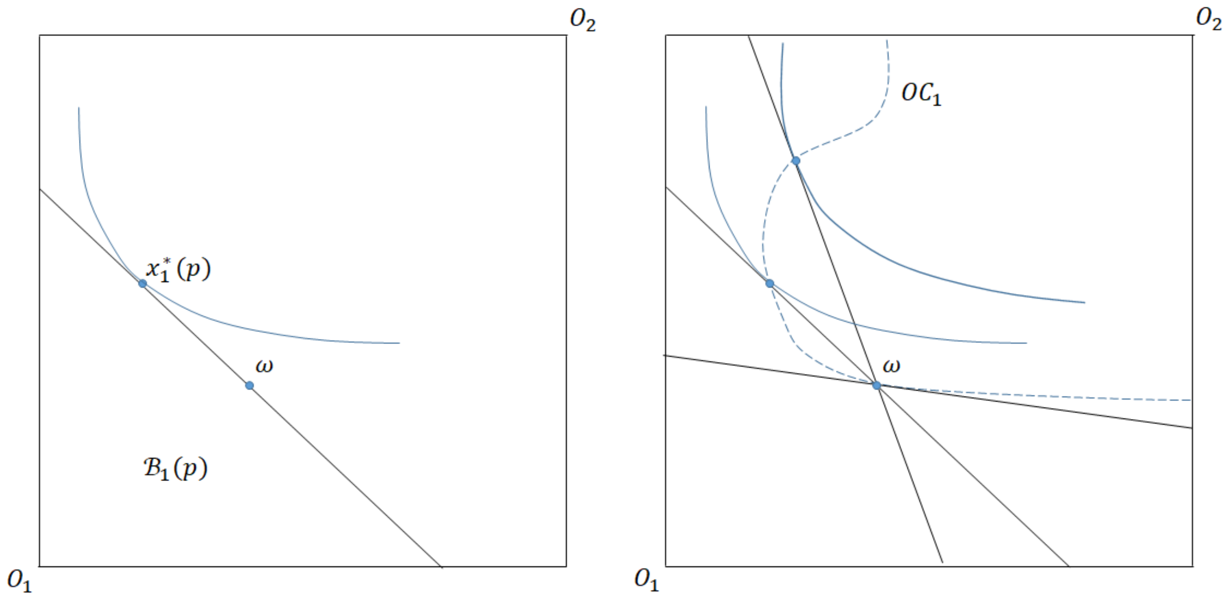


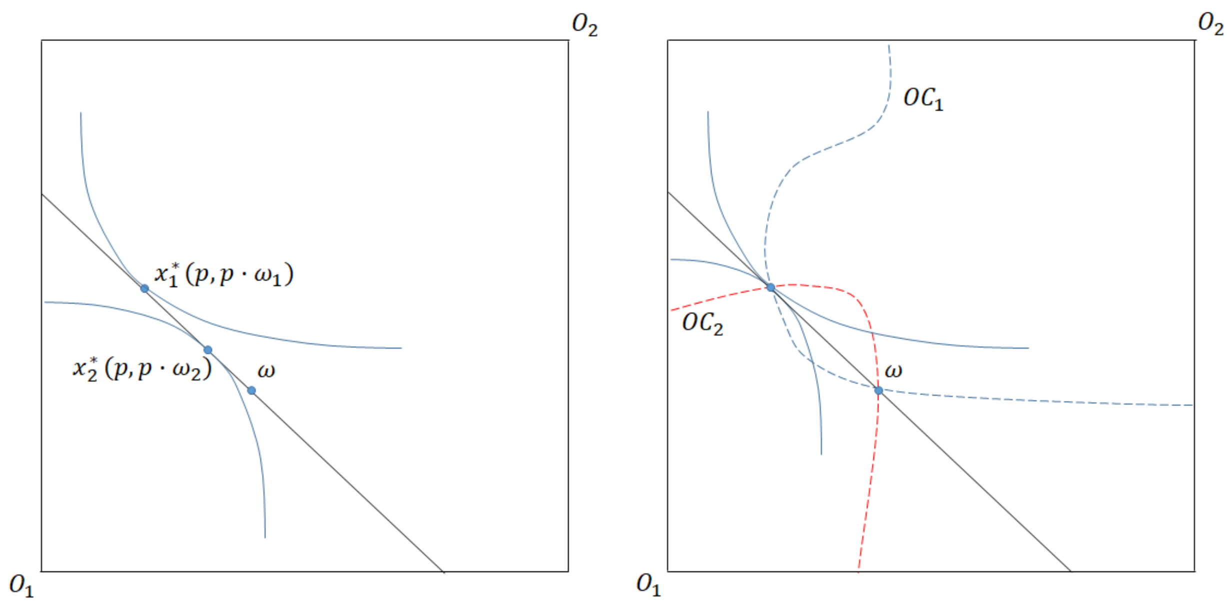
Figure 2(b): prices and budget sets

Figure 2(b) adds prices into the picture and shows that, given any price vector p, the Edgeworth box can be partitioned into the budget sets for the two consumers. Given these prices, consumer 1 can choose any consumption bundle to the bottom left of the diagonal line, and consumer 2 can choose any consumption bundle to the upper right of the diagonal line. Given these prices, Figure 3(a) shows that consumer 1 will optimally choose bundle $x_1^* (p, p \cdot \omega_1)$, and Figure 3(a) shows how $x_1^* (p, p \cdot \omega_1)$ varies as the price ratio varies. Note that, in terms of determining consumer 1's optimal choice, the price ratio p_1/p_2 is a sufficient statistic for the price vector p. This is because Marshallian demand correspondences are homogeneous of degree zero in prices (i.e., $x_1^* (p, p \cdot \omega_1) = x_1^* (\lambda p, \lambda p \cdot \omega_1)$ for all $\lambda \in \mathbb{R}_{++}$).The curve traced out in Figure 3(b) is referred to as consumer 1's **offer curve**.



Figures 3(a) and 3(b): Consumer 1's Marshallian demand for a fixed p and her offer curve

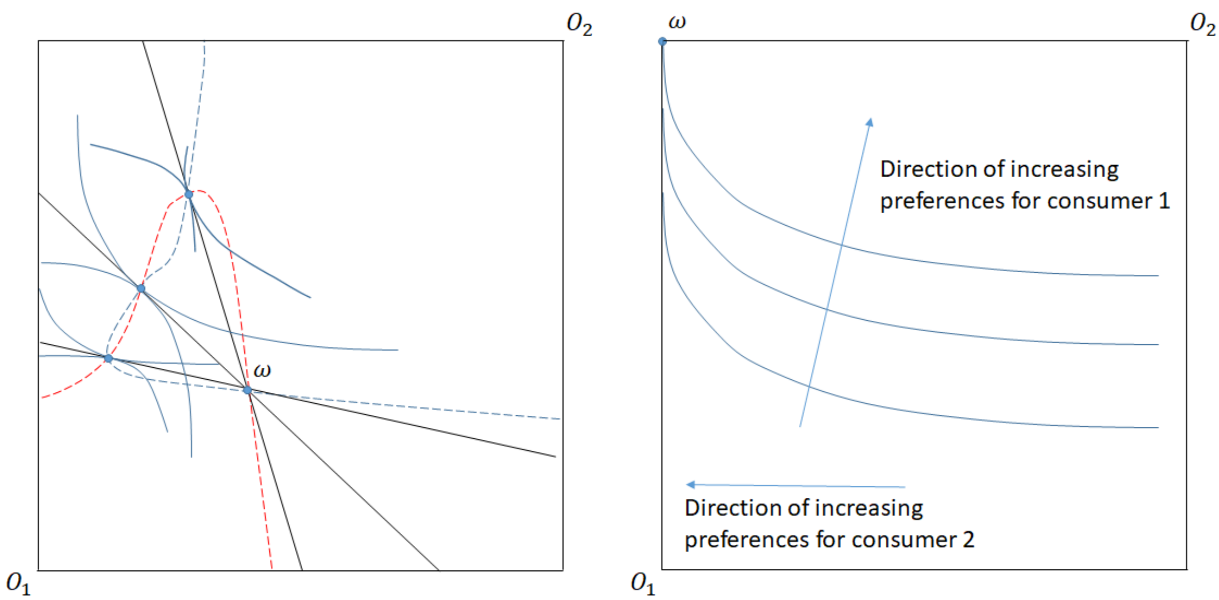
Recall from above that in a Walrasian equilibrium $\left(p^*, (x_i^*)_{i\in\{1,2\}}\right)$, consumer i optimally chooses x_i^* given equilibrium prices p^*. This means that in any Walrasian equilibrium, both consumers' optimal choices lie on their offer curves. Figure 4(a) depicts, for a given price vector p, both consumers' optimal choices. At this price vector, consumer 1 would like to sell $\omega_{1,1} - x_{1,1}^*(p, p\cdot\omega_1)$ units of commodity 1 in exchange for $x_{2,1}^*(p, p\cdot\omega_1) - \omega_{2,1}$ units of commodity 2, and consumer 2 would like to buy $x_{1,2}^*(p, p\cdot\omega_2) - \omega_{1,2}$ units of commodity 1 and sell $\omega_{2,2} - x_{2,2}^*(p, p\cdot\omega_2)$ units of commodity 2. The associated allocation, $(x_1^*(p, p\cdot\omega_1), x_2^*(p, p\cdot\omega_2))$, is not a Walrasian equilibrium allocation, since consumer 1 would like to sell more units of commodity 1 than consumer 2 would like to buy, so the market for commodity 1 does not clear: $\omega_{1,1} - x^*_{1,1}(p, p \cdot \omega_1) > x^*_{1,2}(p, p \cdot \omega_2) - \omega_{1,2}$.



Figures 4(a) and 4(b): disequilibrium and equilibrium allocations

It should be clear from the above argument, then, that any Walrasian equilibrium allocation has to occur at a point where both consumers' offer curves intersect. Figure 4(b) illustrates such a point. The upper-left point at which the two offer curves intersect is a Walrasian equilibrium allocation, and the price vector that ensures both players optimally choose the associated consumption bundles is a Walrasian equilibrium price vector. This example also illustrates that, relative to the Walrasian equilibrium allocation, there are no other feasible allocations that can make one of consumers better off without hurting the other consumer. The Walrasian equilibrium allocation is therefore Pareto optimal. Note that the offer curves also intersect at the endowment point, but the endowment point is not a Walrasian equilibrium allocation in this particular example—why?

The Edgeworth box is also useful for illustrating why there may be multiple Walrasian equilibria and why a Walrasian equilibrium might fail to exist. Figure 5(a) illustrates a situation in which there are multiple Walrasian equilibria. Here, the two consumers' offer curves intersect multiple times. Exercise 2 asks you to solve for the set of Walrasian equilibria in another example in which there are multiple equilibria.



Figures 5(a) and 5(b): multiple Walrasian equilibria and no Walrasian equilibria

Figure 5(b) illustrates a situation in which there are no Walrasian equilibria. In the example, consumer 1 is endowed with no units of commodity 1 and with all the units of commodity 2. Consumer 2 is endowed with all the units of commodity 1 and with no units of commodity 2. Consumer 2 cares only about her consumption of commodity 1, and consumer 2 cares about both her consumption of commodity 1 and her consumption of commodity 2. Moreover, consumer 1's marginal utility of consuming the first unit of commodity 1 is infinite, and her marginal utility of consuming commodity 2 is strictly positive. For any prices p with $p_1 > 0$, the market for commodity 1 cannot clear, since consumer 2 will always choose $x^*_{1,2}(p, p \cdot \omega_2) = \omega_{1,2}$, and consumer 1 will always choose $x^*_{1,1}(p, p \cdot \omega_1) > 0$, unless $p_2 = 0$. And if $p_2 = 0$, then $x^*_{2,1}(p, p \cdot \omega_1) = +\infty$, so the market for commodity 2 cannot clear. This example illustrates why things can go awry when assumption $(A4)$ is not satisfied. These examples tell us that the answers to the following two important questions is "no":

(a) is there always a Walrasian equilibrium? (b) if there is a Walrasian equilibrium, is it unique?

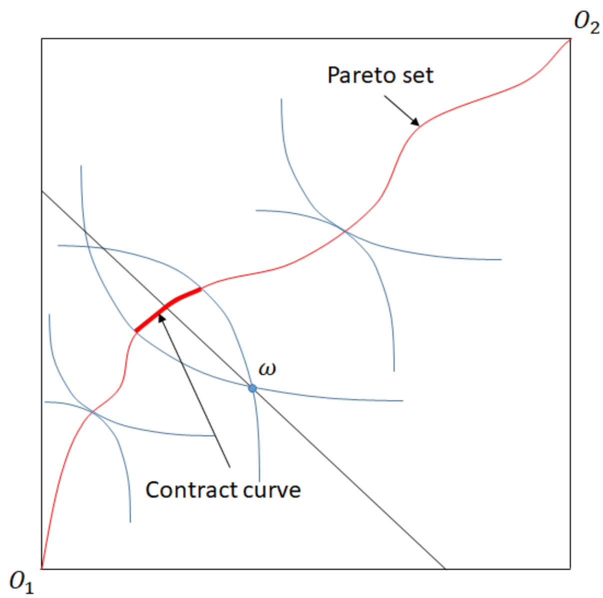


Figure 6: Pareto-optimal allocations and the contract curve

Finally, Figure 6 shows that we can use the Edgeworth box to illustrate the entire set of Pareto-optimal allocations. The **Pareto set** is the set of all feasible allocations for which making one consumer better off necessarily means making the other consumer worse off. It also illustrates the **contract curve**, which is the set of Pareto-optimal allocations that both players prefer to the endowment. If the two consumers were to negotiate a deal, given their endowments as their outside options, they would likely reach a point on the contract curve. Walrasian equilibrium allocations are typically a small subset of the contract curve, and in particular, they lie on the Pareto set. This result is known as the first welfare theorem, and we will now establish this result.

Exercise 1 (Adapted from MWG 15.B.2). Consider an Edgeworth box economy in which the consumers have Cobb-Douglas utility functions $u_1(x_{1,1}, x_{2,1}) = x_{1,1}^{\alpha} x_{2,1}^{1-\alpha}$ and $u_2(x_{1,2}, x_{2,2}) = x_{1,2}^{\beta} x_{2,2}^{1-\beta}$, where $\alpha, \beta \in (0,1)$. Consumer i's endowments are $(\omega_{1,i}, \omega_{2,i}) >> 0$

for $i = 1, 2$. Solve for the Walrasian equilibrium price ratio and allocation. How do these change as you increase $\omega_{1,1}$? Note: Feel free to avoid writing expressions out as much as possible. For example, if you solve for price, feel free to leave the solutions for demand in terms of the price variable instead of plugging in. For comparative statics, if you can find the sign without having to write it out, that's fine.

Exercise 2 (Adapted from MWG 15.B.6). Compute the Walrasian equilibria for the following Edgeworth box economy (there is more than one Walrasian equilibrium):

$$\begin{aligned} u_1(x_{1,1}, x_{2,1}) &= \left(x_{1,1}^{-2} + \left(\frac{12}{37}\right)^3 x_{2,1}^{-2} \right)^{-1/2}, & \omega_1 = (1, 0), \\ u_2(x_{1,2}, x_{2,2}) &= \left(\left(\frac{12}{37}\right)^3 x_{1,2}^{-2} + x_{2,2}^{-2} \right)^{-1/2}, & \omega_2 = (0, 1). \end{aligned}$$

Exercise 3 (Adapted from MWG 15.B.9). Suppose that in a pure exchange economy, we have two consumers, Alphanse and Betatrix, and two commodities, Perrier and Brie. Alphanse and Betatrix have the utility functions:

$$u_\alpha(x_{p,\alpha}, x_{b,\alpha}) = \min\{x_{p,\alpha}, x_{b,\alpha}\} \text{ and } u_\beta(x_{p,\beta}, x_{b,\beta}) = \min\left\{x_{p,\beta}, (x_{b,\beta})^{1/2}\right\},$$

(where $x_{p,\alpha}$ is Alphanse's consumption of Perrier, and so on). Alphanse starts with an endowment of 30 units of Perrier (and none of Brie); Betatrix starts with 20 units of Brie (and none of Perrier). Neither can consume negative amounts of a commodity. If the two consumers behave as price takers, what is the equilibrium? [Hint: consider the market-clearing condition in the cases when both prices are positive, when only the price of Perrier is positive, and when only the price of Brie is positive.]

Exercise 4. Consider an exchange economy with two consumers. The utility functions and endowments are given by

$$\begin{aligned} u_1(x_{1,1}, x_{2,1}) &= x_{1,1} - \frac{x_{2,1}^{-3}}{3}, & \omega_1 = (K, r) \\ u_2(x_{1,2}, x_{2,2}) &= x_{2,2} - \frac{x_{1,2}^{-3}}{3}, & \omega_2 = (r, K). \end{aligned}$$

Assume that K is sufficiently large so that each consumer can achieve an interior solution to her optimal consumption problem. Note that $p^* = (1, 1)$ is an equilibrium price vector.

(a) For what values of r will there be multiple Walrasian equilibria in this economy? [Hint: first solve for $q = p_y/p_x$ by showing that $rt^4 - t^3 + t - r = 0$, where $t = q^{1/4}$. Note this expression factors as $(t+1)(t-1)(rt^2 - t + r) = 0$.]

(b) For what value of r will $p^* = (1, 3)$ be an equilibrium price vector?

(c) [Optional: algebra intensive] Assume that $K = 10$ and that r takes the value identified in part (b). Find all equilibrium prices and allocations.

(d) [Optional: algebra intensive] Rank the outcomes identified in part (d) in terms of most preferred to least preferred for each consumer.

3 First Welfare Theorem

At the Walrasian equilibria in the examples we just saw, there are no feasible allocations that make both players better off: the Walrasian equilibrium allocation was Pareto optimal. This, it turns out, is a general result and perhaps one of the most important results of GE. This result is known as the first welfare theorem. Before stating and proving the first welfare theorem, we will first establish an intermediate result known as Walras's Law, which is a direct implication of consumer optimization when consumers' preferences are monotonic (or more generally, satisfy local non-satiation).

Lemma 1 (Walras's Law). *Given an economy* $\mathcal{E}$ *and prices* p*, if* $(A2)$ *holds, then* $p \cdot \left(\sum_{i\in\mathcal{I}} x_i(p, p\cdot\omega_i)\right) = p \cdot \left(\sum_{i\in\mathcal{I}} \omega_i\right)$.

Proof of Lemma 1. Since $(A2)$ holds, each consumer will optimally choose to exhaust her budget: $p \cdot x_i(p, p\cdot\omega_i) = p\cdot\omega_i$ for all $i \in \mathcal{I}$. Summing this condition over consumers gives us the expression in the Lemma.■

Note that Walras's Law holds for any set of allocations that are consumer-optimal—the result does not require that the allocation $(x_i(p, p\cdot\omega_i))_{i\in\mathcal{I}}$ is a Walrasian equilibrium allocation.

Exercise 5 (Adapted from MWG 15.B.1). Consider an Edgeworth box economy in which the two consumers' preferences satisfy local nonsatiation. Let $x_{l,i}(p, p\cdot\omega_i)$ be consumer i's demand for commodity l at prices $p = (p_1, p_2)$.

(a) Show that $p_1 \sum_{i\in\mathcal{I}} (x_{1,l} - \omega_{1,l}) + p_2 \sum_{i\in\mathcal{I}} (x_{2,l} - \omega_{2,l}) = 0$ for all prices $p \neq 0$.

(b) Argue that if the market for commodity 1 clears at prices $p^* >> 0$, then so does the market for commodity 2; hence p^* is a Walrasian equilibrium price vector.

We can now prove a version of the first welfare theorem.

Theorem 1 (First Welfare Theorem). *Suppose* $\left(p^*, (x_i^*)_{i\in\mathcal{I}}\right)$ *is a Walrasian equilibrium for the economy* $\mathcal{E}$. *Then if* $(A2)$ *holds, the allocation* $(x_i^*)_{i\in\mathcal{I}}$ *is Pareto optimal.*

Proof of Theorem 1. In order to get a contradiction, suppose the Walrasian equilibrium allocation $(x_i^*)_{i\in\mathcal{I}}$ is not Pareto optimal. Then there is some other feasible allocation $(\hat{x}_i)_{i\in\mathcal{I}}$ for which $u_i(\hat{x}_i) \geq u_i(x_i^*)$ for all $i \in \mathcal{I}$ and $u_{i'}(\hat{x}_{i'}) > u_{i'}(x_{i'}^*)$ for some i'. Since $(x_i^*)_{i\in\mathcal{I}}$ is a Walrasian equilibrium allocation, and consumers' preferences satisfy $(A2)$, by revealed preference, it has to be the case that $p^* \cdot \hat{x}_i \geq p^* \cdot x_i^*$ for all $i \in \mathcal{I}$ and $p^* \cdot \hat{x}_{i'} > p^* \cdot x_{i'}^*$. Summing up over these conditions,

$$p^* \cdot \left(\sum_{i\in\mathcal{I}} \hat{x}_i\right) > p^* \cdot \left(\sum_{i\in\mathcal{I}} x_i^*\right) = p^* \cdot \left(\sum_{i\in I} \omega_i\right),$$

where the equality holds by Lemma 1. Since equilibrium prices p^* are nonnegative (why are they nonnegative?), this inequality implies that there is some commodity l such that $\sum_{i\in\mathcal{I}} \hat{x}_{i,l} > \sum_{i\in\mathcal{I}} \omega_{i,l}$, and therefore $(\hat{x}_i)_{i\in\mathcal{I}}$ is not a feasible allocation.■

The first welfare theorem is a remarkable result because (a) its conclusion is both intellectually important and powerful, (b) its explicit assumptions are quite weak, and (c) it has a simple proof, in the sense that it involves only a couple steps, and each step is completely transparent. Let me comment a bit more on each of these three points.

The first welfare theorem provides a formal statement of a version of Adam Smith's argument that the "invisible hand" of decentralized markets leads selfish consumers to make decisions that lead to socially efficient outcomes. Despite there being no explicit coordination among consumers, the resulting equilibrium allocation is Pareto optimal.

Second, the only explicit assumption we made in order to prove the first welfare theorem was that consumers have monotonic preferences—and even this assumption can be relaxed, as exercise 6 below asks you to show. But in the background, there are several strong and important assumptions. First, we assumed that all consumers face the same prices as each other for all commodities. Second, we assumed that all consumers are price takers—

they take prices as given and understand that their consumption decisions do not affect these prices. Third, there are markets for each commodity, and all consumers can freely participate in each market. Fourth, we assumed that each consumer cares only about her own consumption and not about the consumption of anyone else in the economy—we have therefore ruled out externalities. Finally, we assumed that there are a finite number of commodities and consumers. Exercise 6 asks you to show that when there are an infinite number of commodities and consumers, Walrasian equilibrium allocations need not be Pareto optimal.

Exercise 6 (Adapted from MWG 16.C.3). In this exercise, you are asked to establish the first welfare theorem under a set of assumptions compatible with satiation. First, we will define the appropriate notion of equilibrium. Given an economy $\mathcal{E}$, an allocation $(x_i^*)_{i\in\mathcal{I}}$ and a price vector $p = (p_1, \ldots, p_L)$ constitutes a **price equilibrium with transfers** if there is an assignment of wealth levels $(w_1, \ldots, w_I)$ with $\sum_{i\in\mathcal{I}} w_i = p \cdot \left(\sum_{i\in\mathcal{I}} \omega_i\right)$ such that: (i) consumers optimize: $x_i^*(p, w_i) = x_i^*$ and (ii) markets clear: $\sum_{i\in\mathcal{I}} x_i^* = \sum_{i\in\mathcal{I}} \omega_i$. Suppose that every $\mathcal{X}_i$ is nonempty and convex and that every u_i is strictly convex. Prove the following:

(a) For every consumer i, there is at most one consumption bundle at which she is locally satiated. Such a bundle, if it exists, uniquely maximizes u_i on $\mathcal{X}_i$.

(b) Any price equilibrium with transfers results in a Pareto optimal allocation.

Exercise 7. This exercise illustrates that the importance of the assumption that there are a finite number of commodities for the first welfare theorem. Consider an economy in which there is one physical good, available at infinitely many dates: $t = 1, 2, \ldots$, so there are effectively an infinite number of commodities: the physical good at date 1, the physical good at date 2, and so on. One consumer (or "generation") is born at each date $t = 1, 2, \ldots$, and lives and consumes at dates t and $t+1$ ("young" and "old"). We will refer to the consumer born in date t as consumer t. There is also one old consumer alive at date $t = 1$ (call her consumer 0), and she is endowed with zero units of the good. Each consumer is endowed with one unit of the good when she is young, no units of the good when she is old, and no storage is possible. Consumption in each period is non-negative, and each consumer t's preferences over consumption is given by $u_t(x_{t,t}, x_{t+1,t}) = u(x_{t,t}) + u(x_{t+1,t})$, where u is smooth, increasing, and strictly concave, with $u'(0) < \infty$.

(a) Show that there is a Walrasian equilibrium in which each consumer consumes her endowment and gets utility $u(1) + u(0)$.

(b) Show that the above Walrasian equilibrium is unique.

(c) Show that the above Walrasian equilibrium allocation is not Pareto optimal. In other words, construct a feasible allocation that is strictly better for each consumer.

I want to conclude this section with a couple comments on the simplicity of the proof of the first welfare theorem. First, the theorem itself presents a partial characterization of equilibrium allocations. To prove the statement, we did not need to solve explicitly for a Walrasian equilibrium and show that it is Pareto optimal. Instead, we described properties that all Walrasian equilibrium allocations must satisfy. Second, the statement itself is a conditional statement. It is a statement of the form "if $(x_i^*)_{i\in\mathcal{I}}$ is a Walrasian equilibrium, then $(x_i^*)_{i\in\mathcal{I}}$ is Pareto optimal." This conditional statement dodges the question of whether there is in fact a Walrasian equilibrium—we showed above that there does not always exist a Walrasian equilibrium, and we will spend some time next week providing conditions under which a Walrasian equilibrium in fact exists. Finally, the proof is a proof by contradiction, and it effectively takes the form of "if this Walrasian equilibrium allocation was not Pareto-optimal, then stuff doesn't add up." While elegant, the proof itself provides little insight into *why* the first welfare theorem holds. We will spend a little more time discussing the "why" in week 3.

4 Second Welfare Theorem

The first welfare theorem establishes that Walrasian equilibrium allocations are Pareto optimal. The second welfare theorem in some sense establishes a converse. It says that, under some assumptions, any Pareto optimal allocation can be "decentralized" as a Walrasian equilibrium allocation, given the correct prices and endowments.

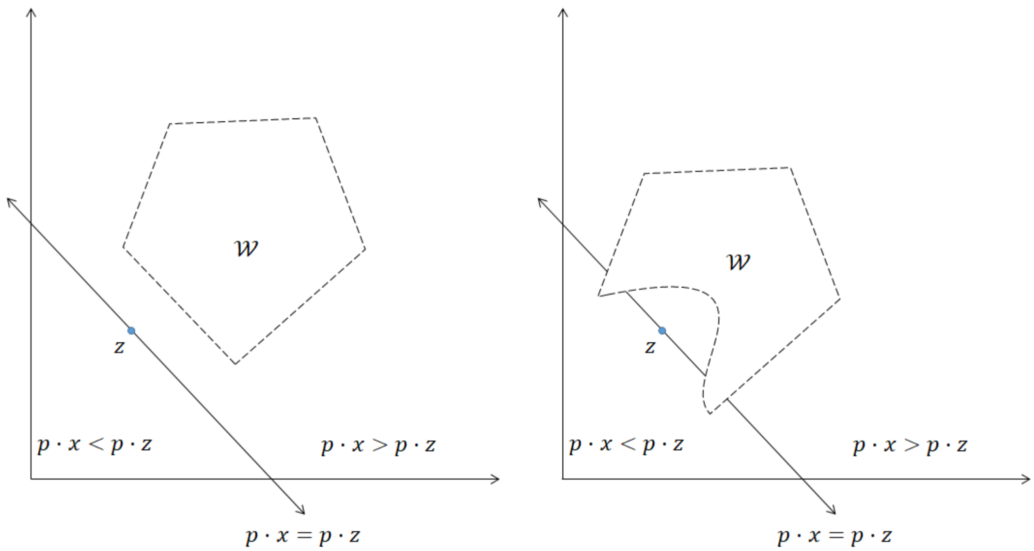
Theorem 2 (Second Welfare Theorem). *Let $\mathcal{E}$ be an economy that satisfies $(A1)-(A4)$ and $X_i = R_+^L$. If $(\omega_i)_{i\in\mathcal{I}}$ is Pareto optimal, then there exists a price vector $p \in R_+^L$ such that $\left(p, (\omega_i)_{i\in\mathcal{I}}\right)$ is a Walrasian equilibrium for $\mathcal{E}$.*

Before proving the second welfare theorem, we will state a version of an important theorem in convex analysis, which is used in the key step of the proof of the second welfare theorem.

Lemma 2 (Separating Hyperplane Theorem). *If $\mathcal{W} \subseteq R^n$ is an open convex set, and $z \notin \mathcal{W}$ is a point not in $\mathcal{W}$, then there exists a vector $p \neq 0$ and such that $p \cdot x \geq p \cdot z$ for all $x \in cl(\mathcal{W})$.*

Figure 7(a) illustrates this version of the separating hyperplane theorem in two dimensions. The set $\mathcal{W}$ is open and convex, and $z \notin \mathcal{W}$. The point z is on a line (which is a hyperplane in a two-dimensional space) characterized by the equation $p \cdot x = p \cdot z$ (i.e., p is the normal vector to the line). All the points to the upper right of that line satisfy $p \cdot x > p \cdot z$, and all the points to the lower left of that line satisfy $p \cdot x < p \cdot z$. And in particular, $\mathcal{W}$ is fully to the upper right of this line. In the case illustrated in Figure 7(a), there are of course many other separating hyperplanes satisfying $p \cdot x \geq p \cdot z$ for all $x \in \mathcal{W}$ corresponding to differently sloped lines going through z but not intersecting $\mathcal{W}$. Figure 7(b) shows why the assumption that $\mathcal{W}$ is a convex set is important for this result. If $z \notin \mathcal{W}$, but $z \in conv(\mathcal{W})$,[3] then there is no vector $p \neq 0$ for which $p \cdot x \geq p \cdot z$ for all $x \in \mathcal{W}$. Exercise 8 asks you to prove a stronger version of the separating hyperplane theorem, which shows that any two disjoint convex sets can be separated by a hyperplane.

[3]The set $conv(\mathcal{W})$ is defined to be the smallest convex set containing $\mathcal{W}$. In two dimensions, you can visualize $conv(\mathcal{W})$ by taking $\mathcal{W}$ and putting a rubber band around it.



Figures 7(a) and 7(b): separating hyperplane theorem and convexity.

The idea of the second welfare theorem is to show that, if the endowment $(\omega_i)_{i\in\mathcal{I}}$ is Pareto optimal, we can always find a price vector that separates the set of allocations preferred by all consumers in the economy from $(\omega_i)_{i\in\mathcal{I}}$ and therefore show that $\left(p, (\omega_i)_{i\in\mathcal{I}}\right)$ is a Walrasian equilibrium.

Proof of Theorem 2. By the statement of the theorem, $(\omega_i)_{i\in\mathcal{I}}$ is Pareto optimal. Let us define the set of aggregate consumption bundles that can be allocated in such a way among consumers to make them all strictly better off than under $(\omega_i)_{i\in\mathcal{I}}$. To do so, define the set of consumption bundles that consumer i prefers to ω_i:

$$\mathcal{A}_i = \left\{a \in \mathbb{R}^L : a + \omega_i \geq 0 \text{ and } u_i\left(a + \omega_i\right) > u_i\left(\omega_i\right)\right\}.$$

Since u_i is concave, the set $\mathcal{A}_i$ is convex. The Minkowski sum of the sets $\mathcal{A}_i$ is therefore also a convex set.[4] That is, if we define

$$\mathcal{A} = \sum_{i \in \mathcal{I}} \mathcal{A}_i = \left\{ a \in \mathbb{R}^L : \exists a_1 \in \mathcal{A}_1, \ldots, \exists a_I \in \mathcal{A}_I \text{ with } a = \sum_{i \in \mathcal{I}} a_i \right\},$$

then $\mathcal{A}$ is a convex set. The set $\mathcal{A}$ does not contain the 0 vector because $(\omega_i)_{i \in \mathcal{I}}$ is Pareto optimal. To see why this is the case, note that if $0 \in \mathcal{A}$, then there would exist $(a_i)_{i \in \mathcal{I}}$ with $\sum_{i \in \mathcal{I}} a_i = 0$ and $u_i(a_i + \omega_i) > u_i(\omega_i)$ for all i. That is, we could essentially just reallocate the endowment $(\omega_i)_{i \in \mathcal{I}}$ among the I consumers and make them all strictly better off, but that would contradict the assumption that $(\omega_i)_{i \in \mathcal{I}}$ is Pareto optimal.

Next, by Lemma 2, there is some price vector $p^* \neq 0$ such that $p^* \cdot a \geq 0$ for all $a \in cl(\mathcal{A})$. Moreover, each of the prices $p_l^* \geq 0$. To see why, suppose $p_l^* < 0$ for some l. Take some a for which a_l is arbitrarily large and all other $a_{l'}$ are arbitrarily small but positive. By the monotonicity of consumer preferences, $a \in \mathcal{A}$, but if a is chosen this way, then $p \cdot a < 0$. We therefore have that $p^* > 0$ (i.e., $p_l^* \geq 0$ for all $l \in \mathcal{L}$ with at least one inequality strict).

We will now show that $\left(p^*, (\omega_i)_{i \in \mathcal{I}}\right)$ is a Walrasian equilibrium. To do so, we need to show that at p^*, consumers optimally consume their endowments and that markets clear. The second condition is immediate. It remains to show that at this p^*, consumers optimally consume their endowments. To do so, suppose there is some $\hat{x}_i \in \mathbb{R}_+^L$ for which $u_i(\hat{x}_i) > u_i(\omega_i)$. We will show that this $\hat{x}_i \notin \mathcal{B}_i(p^*)$. By the definition of $\mathcal{A}$, the allocation $(x_i)_{i \in \mathcal{I}} - (\omega_i)_{i \in \mathcal{I}}$ with $x_i = \hat{x}_i$ and $x_j = \omega_j$ for all $j \neq i$, is in $cl(\mathcal{A})$. By the definition of p^*, we necessarily have that $p^* \cdot (\hat{x}_i - \omega_i) + p^* \cdot \sum_{j \neq i} (\omega_j - \omega_j) \geq 0$, which implies that $p^* \cdot \hat{x}_i \geq p^* \cdot \omega_i > 0$, where this last inequality holds because of Assumption $(A4)$ that all consumers have positive endowments of all commodities.

We are not yet done, because we have to show that this last inequality is strict. This is where continuity of preferences (Assumption $(A1)$) comes into the picture. Since $u_i(\hat{x}_i) >$

[4] The Minkowski sum of two sets $\mathcal{A}$ and $\mathcal{B}$ is just the set of vectors x that can be written as the sum of vectors $x = a + b$ for which $a \in \mathcal{A}$ and $b \in \mathcal{B}$. The closest visual analog to thinking about the Minkowski sum of sets in two dimensions is the way the clone stamp tool in Photoshop works if you are familiar with it.

$u_i(\omega_i)$, this implies that for λ just less than 1, $u_i(\lambda \hat{x}_i) > u_i(\omega_i)$, which in turn implies that $\lambda p^* \cdot \hat{x}_i \geq p^* \cdot \omega_i > 0$. This cannot be the case if $p^* \cdot \hat{x}_i = p^* \cdot \omega_i$, so we must therefore have that $p^* \cdot \hat{x}_i > p^* \cdot \omega_i$ and hence $\hat{x}_i \notin \mathcal{B}_i(p^*)$—that is, any allocation preferred by consumer i to her endowment is unaffordable, and hence her optimal consumption bundle is her endowment.■

The second welfare theorem does not show that every Pareto optimal allocation is a Walrasian equilibrium given a *particular* endowment. Instead, it says that if we were to start from a particular endowment $(\omega_i)_{i \in \mathcal{I}}$, and an allocation $(x_i)_{i \in \mathcal{I}}$ is Pareto optimal, then we could reallocate consumers' endowments in such a way that $(x_i)_{i \in \mathcal{I}}$ is a Walrasian equilibrium allocation. The version of the theorem that we just proved carries out this exercise using a particularly stark reallocation of endowments (i.e., it just sets $(\omega_i)_{i \in \mathcal{I}} = (x_i)_{i \in \mathcal{I}}$). There are versions of the theorem that involve carrying out lump-sum transfers of wealth rather than directly moving around endowments. As you might expect, decentralizing a particular Pareto-optimal allocation in practice potentially requires large-scale redistribution of wealth. I view the result more as establishing an equivalence between Walrasian equilibria and Pareto-optimal allocations rather than as a practical guide for figuring out how to achieve a particular distribution of consumption in society.

It is worth a reminder that convexity of consumers' preferences was critical in establishing the result that $\mathcal{A}$ was a convex set, which in turn is required for using the separating hyperplane theorem. Figure 8 shows an example where the conclusion of the second welfare theorem fails if consumers' preferences are not convex.

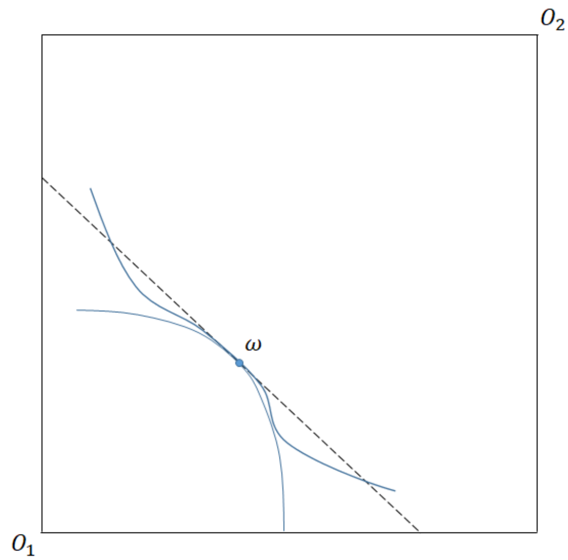


Figure 8: Non-convex preferences

In this figure, the endowment is a Pareto-optimal allocation, since consumer 1's and consumer 2's better-than sets are separated. But there are no prices that can make it optimal for consumer 1 to consume ω_1.

Nevertheless, a version of the second welfare theorem continues to hold when consumers do not have convex preferences if you replicate the economy a large number of times. Think of the 2-consumer economy as being a metaphor for a large economy with two *types* of consumers: type-1 consumers have preferences u_1 and endowments ω_1, and type-2 consumers have preferences u_2 and endowments ω_2. If we replicate the economy a large number of times, so that there are N type-1 consumers and N type-2 consumers, where N is large, then we can support ω as a Walrasian equilibrium allocation, at least on average. This result follows from an application of the Shapley-Folkman lemma, which roughly says that the Minkowski average of sets converges to the convex hull of that set. You don't need to know the math behind this result, but it is a useful result to be aware of. Figure 9 illustrates a replication economy for the economy described in Figure 8. It shows that there may be a p for which there is an $x_1 \in x_1(p, p \cdot \omega_1)$ and an $x_1' \in x_1(p, p \cdot \omega_1)$, so that if we allocate a fraction λ of type-1 consumers to consume x_1 and a fraction $1-\lambda$ of type-1 consumers to consume x_1', on average they are consuming ω_1: $\lambda x_1 + (1-\lambda) x_1' = \omega_1$.

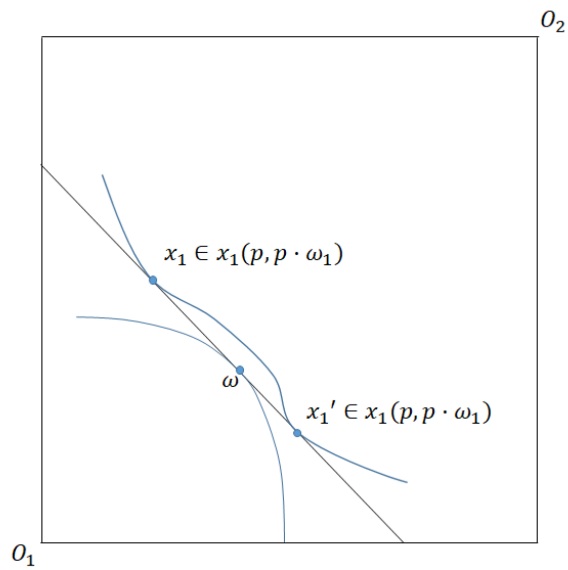


Figure 9: Replication economy

This figure illustrates the idea that large numbers "convexifies" the economy. There is a recurring theme throughout general equilibrium theory that many of the pathologies that arise seem to "go away" in sufficiently large economies. Nonconvexities seem esoteric, since we usually think of consumers' preferences as having diminishing marginal utility and preferences for variety. Nonconvexities become especially relevant when we think of firms, though. When there are fixed costs, for example, the firm analogue of consumers' "better-than" sets are not convex, since the set of production levels better than "not even breaking even" can include both "shut down" and "produce, but at a much larger scale."

Finally, we made use of Assumption $(A4)$ in a somewhat opaque way in the proof. What Assumption $(A4)$ rules out is cases like the one illustrated in Figure 5(b) in which there were no Walrasian equilibria. The failure of equilibrium existence illustrated in Figure 5(b) arises because of a sort of "division by zero" problem: supporting the endowment as an equilibrium allocation would have required consumer 2 to buy only a finite amount of a commodity with a zero price when she has zero wealth.

Exercise 8. This question is intended to guide you through a proof of the separating hyperplane theorem. This is more of an exercise in math than in economics, so feel free to skip to the next step if you get stuck.

(a) Prove that if $y \in \mathbb{R}^N$ and $\mathcal{C} \subseteq \mathbb{R}^N$ is closed, then there exists a point $z \in \mathcal{C}$ such that $||z - y|| \leq ||x - y||$ for all $x \in \mathcal{C}$. That is, there exists a point in $\mathcal{C}$ that is closest to y. (You may assume that $||\cdot||$ is the Euclidean norm.) Hint: use the Weierstrass extreme value theorem—if f is a real-valued and continuous function on domain $\mathcal{S}$, and $\mathcal{S}$ is compact and non-empty, then there exists x such that $f(x) \geq f(y)$ for all $y \in \mathcal{S}$.

(b) Suppose further that $\mathcal{C} \subseteq \mathbb{R}^N$ is convex, and note from above that if $y \notin \mathcal{C}$, then there exists $z \in \mathcal{C}$ that is closest to y. Let $x \in \mathcal{C}$ with $x \neq z$.

(i) Show that $(y - z) \cdot z \geq (y - z) \cdot x$. Hint: consider $||y - (z + t(x - z))||$ for $t \in [0, 1]$, the distance between y and a convex combination of x and z.

(ii) Use the above result to show that for all $x \in \mathcal{C}$, $(y - z) \cdot y > (y - z) \cdot x$.

(iii) Explain how this is a special case of the separating hyperplane theorem, which states that for any disjoint convex sets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^N$, there exists nonzero $p \in \mathbb{R}^N$ such that $p \cdot u \geq p \cdot v$ for any $u \in \mathcal{A}$ and $v \in \mathcal{B}$.

(iv) Use the result of (ii) to deduce the separating hyperplane theorem. Hint: consider $y = 0$ and $\mathcal{C} = \mathcal{A} - \mathcal{B} = \{u - v : u \in \mathcal{A}, v \in \mathcal{B}\}$.

5 Characterizing Pareto-Optimal Allocations

The welfare theorems provide a tight connection between the set of Pareto optimal allocations and the set of Walrasian equilibrium allocations. This section will provide a short note on how to find Pareto optimal allocations in particularly well-behaved environments. Define the **utility possibility set**

$$\mathcal{U} = \left\{(u_1, \ldots, u_I) \in \mathbb{R}^I : \text{there is a feasible allocation } (x_i)_{i \in \mathcal{I}} \text{ with } u_i(x_i) \geq u_i \text{ for all } i\right\}.$$

If the sets $\mathcal{X}_i$ are convex sets and consumers' preferences are concave, then $\mathcal{U}$ is a convex set. When this is the case, the problem of finding Pareto-optimal allocations can be reduced to the problem of solving **Pareto problems** of the form

$$\max_{u \in U} \lambda \cdot u$$

for some non-zero vector of **Pareto weights** $\lambda \geq 0$. The objective function of this problem is sometimes called a linear Bergson-Samuelson social welfare function. We will say that u^* is a **Pareto-optimal utility vector** if there is a Pareto-optimal allocation $(x_i)_{i \in \mathcal{I}}$ for which $u_i(x_i) = u_i^*$ for all $i \in \mathcal{I}$. The next theorem establishes the result.

Theorem 3. *If u^* is a solution to the Pareto problem described above for some vector of Pareto weights $\lambda >> 0$, then u^* is a Pareto-optimal utility vector. Conversely, if the utility possibility set $\mathcal{U}$ is convex, then any Pareto-optimal utility vector u^* is a solution to the Pareto problem for some non-zero vector $\lambda \geq 0$.*

Proof of Theorem 3. The first part is immediate: if u^* is not Pareto optimal, then any Pareto-dominating utility vector would give a higher value in the Pareto problem for any Pareto weight vector $\lambda >> 0$.

The second part of the theorem makes use of the *supporting hyperplane theorem*, which says that a convex set can be separated from any point outside its interior (see Section M.G of the mathematical appendix of MWG). If u^* is a Pareto-optimal utility vector, then it lies on the boundary of $\mathcal{U}$, so by the supporting hyperplane theorem, there exists $\lambda \neq 0$ such that $\lambda \cdot u^* \geq \lambda \cdot u$ for all $u \in \mathcal{U}$. Further, the Pareto weights satisfy $\lambda \geq 0$, since if $\lambda_i < 0$ for some i, then $\lambda \cdot u^* < \lambda \cdot \tilde{u}$, where for some $K > 0$, $\tilde{u} = \left(u_1^*, \ldots, u_{i-1}^*, u_i^* - K, u_{i+1}^*, \ldots, u_I^*\right) \in \mathcal{U}$. This contradicts the claim that $\lambda \cdot u^* \geq \lambda \cdot u$ for all $u \in \mathcal{U}$, so it must be the case that $\lambda \geq 0$.■

The theorem shows that when the utility possibility set is a convex set, the problem of finding Pareto-optimal allocations boils down to solving a class of Pareto problems. If we further assume that consumers' utility functions are differentiable, then Pareto-optimal allocations can be characterized by taking first-order conditions. For example, suppose utility functions are differentiable with $\nabla u_i(x_i) >> 0$ for all x_i, and we have an interior solution, we can find Pareto-optimal allocations by solving the problem:

$$\max_{(x_i)_{i\in\mathcal{I}}} \sum_{i\in\mathcal{I}} \lambda_i u_i(x_i)$$

subject to feasibility for each commodity:

$$\sum_{i\in\mathcal{I}} x_{l,i} \leq \sum_{i\in\mathcal{I}} \omega_{l,i} \text{ for all } l \in \mathcal{L}.$$

Then one can use the Kuhn-Tucker theorem to verify that any Pareto-optimal allocation $(x_i)_{i\in\mathcal{I}}$ with $x_i >> 0$ for all $i \in \mathcal{I}$ must satisfy

$$\frac{\partial u_i/\partial x_{l,i}}{\partial u_i/\partial x_{l',i}} = \frac{\partial u_{i'}/\partial x_{l,i'}}{\partial u_{i'}/\partial x_{l',i'}} = \frac{\mu_l}{\mu_{l'}} \text{ for all } i, i', l, l'$$

for some $\mu_l, \mu_{l'} > 0$. This condition says that the marginal rate of substitution between any two commodities must be equalized across consumers in any Pareto-optimal allocation. If this condition failed, there would be a Pareto-improving exchange of commodities l and l' between consumers i and i'. The values μ_l corresponds to the Lagrange multiplier on the commodity-l feasibility constraint $\sum_{i\in\mathcal{I}} x_{l,i} = \sum_{i\in\mathcal{I}} \omega_{l,i}$.

As an illustration of the second welfare theorem, given a Pareto-optimal allocation $(x_i)_{i\in\mathcal{I}}$ that satisfies the optimality conditions above, if you set $p_l = \mu_l$ for all $l \in \mathcal{L}$, then $\left(p, (x_i)_{i\in\mathcal{I}}\right)$ is a Walrasian equilibrium of the economy $\mathcal{E} = \left((u_i)_{i\in\mathcal{I}}, (x_i)_{i\in\mathcal{I}}\right)$. This point is illustrated in Figure 4(b). In that figure, at the Walrasian equilibrium allocation, consumers' marginal rates of substitution across the two commodities were equalized. Moreover, these marginal rates of substitution were also equal to the price ratio that corresponded to the Walrasian equilibrium (and given that price ratio, moving the endowment along the boundary of consumers' budget sets does not change their ultimate consumption choices, so the same price vector would also be an equilibrium price vector if we just set consumers' endowments equal to their Walrasian equilibrium allocations).

Exercise 9 (Adapted from MWG 16.C.4). Suppose that for each consumer, there is a "happiness function" depending on her own consumption only, given by $u(x_i)$. Every consumer's utility depends positively on her own and everyone else's "happiness" according to the utility function

$$U_i(x_1, \dots, x_l) = U_i(u_1(x_1), \dots, u_l(x_l)).$$

Show that if $x = (x_1, \dots, x_l)$ is Pareto optimal relative to the $U_i(\cdot)$'s, then $x = (x_1, \dots, x_l)$ is also a Pareto optimum relative to the u_i's. Does this mean a community of altruists can use competitive markets to attain Pareto optima? Does your argument depend on the concavity of the u_i's or the U_i's?

6 Existence of Walrasian Equilibrium

Last week, we focused on the normative, efficiency properties of Walrasian equilibria. This week, we will focus on a couple positive properties. In particular, we will begin by asking what seems like a straightforward question: does a Walrasian equilibrium exist? And then we will ask a few other important questions relating to equilibrium uniqueness, equilibrium stability, and the comparative statics of Walrasian equilibria.

The question of whether a Walrasian equilibrium exists really boils down to: under what conditions on preferences and endowments does a Walrasian equilibrium exist? We know from the example in Figure 5(b) from last week that a Walrasian equilibrium does not always exist. And we know from the second welfare theorem that when assumptions $(A1) - (A4)$ are satisfied, then if the endowment is a Pareto-optimal allocation, there is a Walrasian equilibrium for which it is the equilibrium allocation. In some sense, the second welfare theorem provides a bit of a mundane answer to the existence question, since it provides conditions under which no trade is optimal for each consumer. The more interesting question is the more difficult one: when is a Walrasian equilibrium guaranteed to exist if the endowment itself is not already Pareto optimal? That is, when is there a Walrasian equilibrium that actually involves trade?

This was an open question ever since Walras's formulation of the GE model in the 1870's until Arrow, Debreu, and McKenzie produced the first rigorous existence proofs in the 1950's. The basic question is, given aggregate demand functions $\sum_{i\in\mathcal{I}} x_{l,i}(p, p\cdot\omega_i)$ for each commodity, when does there exist a price vector p^* such that $\sum_{i\in\mathcal{I}} x_{l,i}(p^*, p^*\cdot\omega_i) = \sum_{i\in\mathcal{I}} \omega_{l,i}$ for all $l \in \mathcal{L}$? Early arguments amounted to just counting up the number of equations and unknowns, but these approaches were not satisfactory, since it would not be clear what would happen if the solution to the equations involved negative prices or quantities. The breakthrough came in the 1950's when Arrow and Debreu (1954) proved the following existence result.

Theorem 4 (Existence of Walrasian Equilibrium). *Given an economy $\mathcal{E}$ satisfying $(A1)-(A4)$, there exists a Walrasian equilibrium $\left(p^*, (x_i^*)_{i\in\mathcal{I}}\right)$.*

The key insight in the 1950's was to reframe the Walrasian equilibrium existence question as a *fixed-point* question, following John Nash's (1951) proof of the existence of Nash equilibrium using a related approach. A *fixed point* of a correspondence $f : Z \rightrightarrows Z$ is a point z such that $z \in f(z)$, and fixed-point theorems provide fairly general conditions under which functions or correspondences have fixed points.

The important step in making use of the general-purpose technology of fixed-point theorems is to figure out how to map the equilibrium existence question into the question of whether a suitably chosen correspondence has a fixed point: it is about choosing the right correspondence. Suppose the correspondence f maps an allocation $(x_i)_{i\in\mathcal{I}}$ and a price vector p into a new allocation $(x_i')_{i\in\mathcal{I}}$ and price vector p', where the new allocation is the set of optimal choices for consumers given the price vector p, and the new price vector p' is one that raises the prices of over-demanded goods and lowers the price of under-demanded goods under the allocation $(x_i)_{i\in\mathcal{I}}$ and otherwise does not change prices. Then a fixed point of f will be a Walrasian equilibrium, so if we can show that f satisfies the conditions required for a fixed-point theorem to apply, then we can conclude that a Walrasian equilibrium exists.

6.1 Two-Commodity Intuitive Sketch

We will first go through an intuitive argument for equilibrium existence in the special case of two commodities, and then we will go through the more general result described in Theorem 4. The argument in the two-commodity case will also develop some tools that will be useful when we talk about uniqueness and stability of Walrasian equilibrium. For this part, we will strengthen the monotonicity condition $(A2)$ to a strong monotonicity condition $(A2')$.

Assumption A2' (strong monotonicity). For all consumers $i \in \mathcal{I}$, u_i is strictly increasing: $u_i(x_i') > u_i(x_i)$ whenever $x_{l,i}' \geq x_{l,i}$ for all $l \in \mathcal{L}$ with at least one inequality strict.

As a starting point, we are going to introduce the idea of an excess demand function for an economy $\mathcal{E} = (u_i, \omega_i)_{i \in \mathcal{I}}$. The **excess demand function for consumer** i is $z_i(p) = x_i(p, p \cdot \omega_i) - \omega_i$. The **aggregate excess demand function** is the sum of consumers' excess demand functions $z(p) = \sum_{i \in \mathcal{I}} z_i(p)$. It should be clear from the definition of the aggregate excess demand function that if there is a p^* that satisfies $z(p^*) = 0$, then $\left(p^*, (x_i^*)_{i \in \mathcal{I}}\right)$ with $x_i^* = x_i(p^*, p^* \cdot \omega_i)$ is a Walrasian equilibrium. The way x_i^* is defined, it is clear that at p^*, x_i^* is consumer i's optimal consumption bundle. Moreover, if $z(p^*) = 0$, then markets clear for each commodity. In this case, proving existence boils down to establishing that a solution to $z(p) = 0$ exists given assumptions $(A1), (A2'), (A3)$, and $(A4)$. The aggregate excess demand function inherits many of the properties of Marshallian demand functions, as the next lemma illustrates.

Lemma 3. *Suppose $\mathcal{E}$ satisfies $(A1), (A2'), (A3)$,$(A4)$, and $\mathcal{X}_i = R_+^L$ for all i. Then the aggregate excess demand function satisfies:*

(i) z is continuous;

(ii) z is homogeneous of degree zero;

(iii) $p \cdot z(p) = 0$ for all p (Walras's Law);

(iv) there is some $Z > 0$ such that $z_l(p) > -Z$ for every $l \in \mathcal{L}$ and for every p;

(v) *if* $p^n \to p$, *where* $p \neq 0$ *and* $p_l = 0$ *for some* l, *then* $\max\{z_1(p^n), \ldots, z_L(p^n)\} \to \infty$.

The first property is something that was assumed in Economics 2010a, but it is straightforward to show that it follows from assumptions $(A1) - (A4)$.[5] The second property is straightforward, and we already proved the third property in week 1. The fourth property follows directly from the assumption that $\mathcal{X}_i = \mathbb{R}^L_+$.

The last property bears some comment. It is saying that as some, but not all, prices go to zero, there must be some consumer whose wealth is not going to zero. Because she has strongly monotone preferences, she must demand more of one of the commodities whose price is going to zero.

To gain intuition for the general existence proof, let us consider the case where there are only two goods in the economy, and let us further assume that consumer preferences are strictly concave, so that $x_i(p, p \cdot \omega_i)$ is a singleton for all p (we will allow for $x_i(p, p \cdot \omega_i)$ to be a correspondence—for there to be multiple optimal allocations for a given consumer at a given price vector—when we prove the general theorem). Our goal is to find a price vector $p = (p_1, p_2)$ for which $z(p) = 0$. Because $z(\cdot)$ is homogeneous of degree zero, we can normalize one of the prices, say p_2, to one. This reduces our search to price vectors of the form $(p_1, 1)$. Moreover, Walras's Law implies that if the market for commodity 1 clears, then so does the market for commodity 2, so it suffices to find a price p_1 such that $z_1(p_1, 1) = 0$.

[5] z is upper hemi-continuous from Berge's maximum theorem, it is non-empty because preferences are continuous, and it is convex-valued because preferences are convex. These properties imply that z is a continuous correspondence.

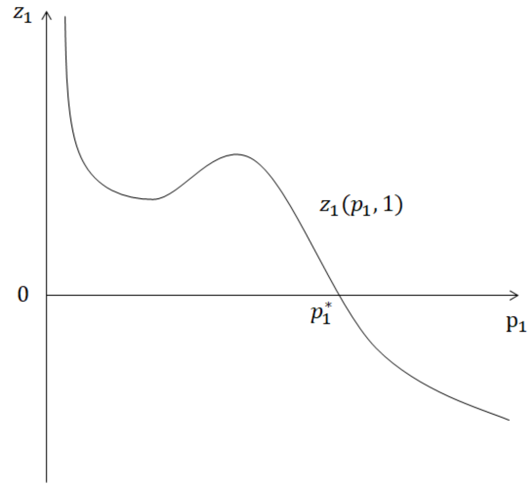


Figure 10: Existence of WE with two commodities

The problem of finding a p_1 such that $z_1(p_1, 1) = 0$ is a one-dimensional problem, so we can just graph it. Figure 10 plots $z_1(p_1, 1)$ as a function of p_1. The figure highlights three important properties of $z_1(p_1, 1)$. First, it is continuous. Second, for p_1 very small, $z_1(p_1, 1) > 0$, and third, for p_1 very large, $z_1(p_1, 1) < 0$. Given these three properties, by the intermediate value theorem—the simplest of fixed-point theorems—there necessarily exists some $p^* = (p_1^*, 1)$ such that $z_1(p^*) = 0$, and a Walrasian equilibrium therefore exists. The subtleties in making this argument are in establishing that $z_1(p_1, 1) > 0$ for p_1 small and $z_1(p_1, 1) < 0$ for p_1 large. The first property follows from condition (v) in the Lemma above. The second property follows because if $p_1 \to \infty$, then each consumer's demand for commodity 1 will converge to something less than her endowment of commodity 1, as continuity and monotonicity of preferences imply she would like to sell at least some of commodity 1 for an unboundedly large amount of commodity 2.

Exercise 10 (Adapted from MWG 17.C.4). Consider a pure exchange economy. The only novelty is that a progressive tax system is instituted according to the following rule: individual wealth is no longer $p \cdot \omega_i$; instead, anyone with wealth above the mean of the population must contribute half of the excess over the mean into a fund, and those below the mean receive a contribution from the fund in proportion to their deficiency below the mean.

(a) For a two-consumer society with endowments $\omega_1 = (1, 2)$ and $\omega_2 = (2, 1)$, write the after-tax wealths of the two consumers as a function of prices.

(b) If the consumer preferences are continuous, strictly convex, and strongly monotone, will the excess demand functions satisfy the conditions required for existence stated in Lemma 3?

6.2 More General Existence Result

Before proving the main existence theorem, we will first remind ourselves of a couple important mathematical theorems that we will be using in the proof. The first is the Kakutani fixed-point theorem, which you used to prove the existence of a Nash equilibrium in Economics 2010a. The second is the maximum theorem or Berge's maximum theorem.

Kakutani Fixed-Point Theorem. *Suppose $\mathcal{Z}$ is a nonempty, compact, convex subset of R^n and that $f : \mathcal{Z} \rightrightarrows \mathcal{Z}$ is a nonempty, convex-valued, and upper hemi-continuous correspondence. Then f has a fixed point.*

Kakutani's fixed-point theorem is a generalization of Brouwer's fixed-point theorem but for set-valued functions. The basic idea of the theorem is that a fixed point is an intersection of the graph of f with the 45° line, and the conditions for the theorem ensure that the graph of f cannot "jump" across the 45° line. In the special case when $n = 1$ and when f is scalar-valued, this theorem boils down to the intermediate-value theorem.

The proof of equilibrium existence is going to make use of the Kakutani fixed-point theorem for an appropriately defined correspondence, and we will need to be able to establish that the correspondence has the properties that are required by the theorem. The following theorem will be useful for establishing these properties.

Berge's Maximum Theorem. *If $f : \mathcal{X} \times \Theta \to R$ is a continuous function, and $C : \Theta \rightrightarrows \mathcal{X}$ is a continuous, compact-valued correspondence, then $V(\theta) = \max\{f(x, \theta) : x \in C(\theta)\}$ is* *continuous in* θ*, and* $X^*(\theta) = \text{argmax}\{f(x,\theta) : x \in C(\theta)\}$ *is non-empty, compact-valued, and upper hemi-continuous.*

The line of proof we will be following is to define a set $\mathcal{Z}$ and a correspondence $f : \mathcal{Z} \rightrightarrows \mathcal{Z}$ that satisfies the conditions of Kakutani fixed-point theorem and whose fixed points are Walrasian equilibria. There are therefore three main questions we will need to answer:

1. What should $\mathcal{Z}$ and f be?

2. Why do the conditions of Kakutani's fixed-point theorem hold?

3. Why do the fixed points of f correspond to Walrasian equilibria?

6.2.1 Step 1: Define $\mathcal{Z}$ and f

To define the set $\mathcal{Z}$, it is convenient to first normalize prices so that they sum to one. Define the normalized price simplex Δ to be the set of associated price vectors: $\Delta \equiv \left\{p \in \mathbb{R}^L_+ : \sum_{l \in \mathcal{L}} p_l = 1\right\}$. Next, for each consumer i, define a non-empty, compact, convex subset of her consumption set that is bounded above by what she could consume if she possessed the entire aggregate endowment: $\mathcal{T}_i = \left\{x_i \in \mathcal{X}_i : x_i \leq 2\sum_{i \in \mathcal{I}} \omega_i\right\} \subset \mathcal{X}_i$. Since each $\mathcal{T}_i$ is a compact, so is the product set $\mathcal{T} = \prod_{i \in \mathcal{I}} \mathcal{T}_i$. Define $\mathcal{Z} \equiv \mathcal{T} \times \Delta$ to be the domain on which we will define the correspondence f.

The correspondence $f : \mathcal{Z} \rightrightarrows \mathcal{Z}$ will map an allocation $(x_i)_{i \in \mathcal{I}}$ and a price vector p to the set of allocations $(x_i')_{i \in \mathcal{I}}$ that are optimal for each consumer given p and a new price vector p' that raises the price of commodities that were over-demanded and lowers the price of commodities that were under-demanded under $(x_i)_{i \in \mathcal{I}}$. The first part of this construction is straightforward. Let $x_i^{\mathcal{T}}(p, p \cdot \omega_i)$ be consumer i's optimal choice over $\mathcal{B}_i(p) \cap \mathcal{T}_i$. Marshallian demand correspondence at prices p:

$$x_i^{\mathcal{T}}(p, p \cdot \omega_i) = \max_{x_i \in \mathcal{B}_i(p) \cap \mathcal{T}_i} u_i(x_i),$$

where recall that $\mathcal{B}_i(p)$ is consumer i's budget set given prices p:

$$\mathcal{B}_i(p) = \{x_i \in \mathcal{X}_i : p \cdot x_i \leq p \cdot \omega_i\}.$$

If we do this for each consumer, we get the product of correspondences $\prod_{i \in \mathcal{I}} x_i^{\mathcal{T}}(p, p \cdot \omega_i) \subset \mathcal{T}$. This takes care of the first part of the construction.

For the second part of the construction, we introduce a fictitious "player" called the Walrasian auctioneer (or the "price player") who chooses a price vector $p \in \mathbb{R}_+^L$ and wants to maximize the value of aggregate excess demand. Let

$$a^*(x) = \underset{\tilde{p} \in \Delta}{\operatorname{argmax}}\, \tilde{p} \cdot \left(\sum_{i \in \mathcal{I}} x_i - \sum_{i \in \mathcal{I}} \omega_i \right),$$

where "a^*" is a pneumonic for "auctioneer." We are now in a position to define the appropriate correspondence $f : \mathcal{Z} \rightrightarrows \mathcal{Z}$ by

$$f(x, p) = \underbrace{\left(\prod_{i \in \mathcal{I}} x_i^{\mathcal{T}}(p, p \cdot \omega_i) \right)}_{\subset \mathcal{T}} \times \underbrace{a^*(x)}_{\subset \Delta}.$$

6.2.2 Step 2: Verify that Kakutani's theorem can be applied

Now that we have defined the set $\mathcal{Z}$ and the correspondence f, we will verify that the conditions of Kakutani's fixed-point theorem hold, so it can be applied. The first set of conditions that needs to be verified is that $\mathcal{Z} = \mathcal{T} \times \Delta$ is a non-empty, compact, convex subset of $\mathbb{R}^n$ for some n. The second set of conditions is on the correspondence f—we need to show that f is a non-empty, convex-valued, and upper hemi-continuous correspondence. Note that the product of non-empty, convex-valued, and upper hemi-continuous correspondences is itself non-empty, convex-valued, and upper hemi-continuous, so this last part requires that we show that each of the correspondences $x_i^{\mathcal{T}}(p, p \cdot \omega_i)$ and $a^*(x)$ satisfy these conditions.

First, note that $\mathcal{Z}$ is a non-empty, compact, and convex subset of $\mathbb{R}^{I \cdot L + L}$ because each

$\mathcal{T}_i$ and Δ are non-empty, compact, and convex subsets of $\mathbb{R}^L$.

Next, a^* is non-empty and convex-valued because Δ is non-empty, compact, and convex, and the Walrasian auctioneer's objective is linear in p and hence continuous. It is upper hemi-continuous by Berge's maximum theorem.

Finally, the function $x_i^{\mathcal{T}}$ is non-empty and convex-valued because $\mathcal{B}_i(p) \cap \mathcal{T}_i$ is non-empty, compact, and convex, and u_i is continuous (guaranteeing x_i^* is non-empty) and concave (guaranteeing x_i^* is convex-valued). These conditions alone are not enough to give us the upper hemi-continuity that we require in order to apply Kakutani's fixed-point theorem, however, because we still have to show that $\mathcal{B}_i(p) \cap \mathcal{T}_i$ is a continuous, compact-valued correspondence.

It is apparent that $\mathcal{B}_i(p) \cap \mathcal{T}_i$ is compact-valued—the involved part is showing that it is a continuous correspondence. To do so, we have to show that it is both upper hemi-continuous in p and lower hemi-continuous in p. Upper hemi-continuity is straightforward, since if $p^n \to p$ and $x_i^n \to x_i$ with $x_i^n \in \mathcal{B}_i(p^n) \cap \mathcal{T}_i$ for all n, then $p^n \cdot x_i^n \leq p^n \cdot \omega_i$ and $x_i^n \leq \sum_{i \in \mathcal{I}} \omega_i$ for all n and therefore this condition holds in the limit as well. Showing that $\mathcal{B}_i(p) \cap \mathcal{T}_i$ is lower hemi-continuous is more involved, and we leave this as an exercise (this is the only part of the existence proof that makes use of assumption $(A4)$).

These arguments establish that $\mathcal{Z}$ and f satisfy the conditions of Kakutani's fixed-point theorem, and therefore f has a fixed point. We now need to show that any such fixed point of f is a Walrasian equilibrium.

6.2.3 Step 3: Show that fixed points of f are Walrasian equilibria

Suppose $(x^*, p^*) \in f(x^*, p^*)$—that is, (x^*, p^*) is a fixed point of f. We need to show that $x_i^* = x_i^{\mathcal{T}}(p^*, p^* \cdot \omega_i)$ is consumer-optimal for each $i \in \mathcal{I}$, and markets clear at prices p^*.

For the first part, because $x_i^{\mathcal{T}}(p^*, p^* \cdot \omega_i) = \max_{x_i \in \mathcal{B}_i(p^*) \cap \mathcal{T}_i} u_i(x_i)$, we need to verify that the resulting solution also solves the relaxed problem $\max_{x_i \in \mathcal{B}_i(p^*)} u_i(x_i)$, which is the problem the consumer actually faces. To do this, first note that since consumers have monotonic preferences, it must be the case that $p^* \cdot \left(\sum_{i \in \mathcal{I}} x_i^*\right) \leq p^* \cdot \left(\sum_{i \in \mathcal{I}} \omega_i\right)$—if we did not have to worry about $x_i \in \mathcal{T}_i$ for each i, this inequality would hold with equality by Walras's Law. Next, since $p^* \in a^*(x^*)$, we have

$$0 \geq p^* \cdot \left(\sum_{i \in \mathcal{I}} x_i^* - \sum_{i \in \mathcal{I}} \omega_i\right) \geq p \cdot \left(\sum_{i \in \mathcal{I}} x_i^* - \sum_{i \in \mathcal{I}} \omega_i\right) \text{ for all } p \in \Delta,$$

so that $\sum_{i \in \mathcal{I}} x_i^* - \sum_{i \in \mathcal{I}} \omega_i \leq 0$ and therefore $x_i^* \leq \sum_{i \in \mathcal{I}} \omega_i$ for all i, so that $x_i^* \in int(\mathcal{T}_i)$. We therefore have that $x_i^* \in \operatorname{argmax}_{x_i \in \mathcal{B}_i(p^*)} u_i(x_i)$ because if there were some $\hat{x}_i \in \mathcal{B}_i(p^*)$ with $u_i(\hat{x}_i) > u_i(x_i^*)$, then for some small λ, $\lambda \hat{x}_i + (1-\lambda) x_i^* \in \mathcal{B}_i(p^*) \cap \mathcal{T}_i$ and by (quasi-)concavity of u_i, $u_i(\lambda \hat{x}_i + (1-\lambda) x_i^*) > u_i(x_i^*)$, which is a contradiction.

We now establish that the market-clearing condition is satisfied. Since x_i^* is consumer-optimal for each i, Walras's Law tells us that $p^* \cdot \left(\sum_{i \in \mathcal{I}} x_i^*\right) = p^* \cdot \left(\sum_{i \in \mathcal{I}} \omega_i\right)$, and in particular, at p^*, the Walrasian auctioneer's value is zero (recall that the auctioneer maximizes $p \cdot \left(\sum_{i \in \mathcal{I}} x_i - \sum_{i \in \mathcal{I}} \omega_i\right)$). If $\sum_{i \in \mathcal{I}} x_{l,i}^* - \sum_{i \in \mathcal{I}} \omega_{l,i}$ were positive for any commodity l, then the auctioneer could set $p_l = 1$ and $p_{l'} = 0$ for all $l' \neq l$ and attain a positive value. This implies that no commodity is over-demanded at the allocation x^*, that is, $\sum_{i \in \mathcal{I}} x_i^* \leq \sum_{i \in \mathcal{I}} \omega_i$.

It remains only to show that this inequality actually holds with equality. By Walras's law, we know that $p^* \cdot \left(\sum_{i \in \mathcal{I}} x_i^*\right) = p^* \left(\sum_{i \in \mathcal{I}} \omega_i\right)$. Since there is no excess demand, this implies that commodity l can be in excess supply only if its price is $p_l^* = 0$. In that case, we can just modify the allocation x^* by giving the entire excess supply of commodity l to some consumer—without loss of generality, let that be consumer 1. This is feasible, and it does not affect consumer 1's utility. Why doesn't it affect her utility? Since her preferences are monotone, giving her more of commodity l cannot decrease her utility. It also cannot increase her utility, because otherwise, she would have chosen the resulting consumption bundle rather than x_i^*, and doing so would have been affordable, because $p_l^* = 0$.

To summarize, either (x^*, p^*) is a Walrasian equilibrium or the allocation resulting from arbitrarily allocating any commodity in excess supply to consumers (along with the price vector p^*) is a Walrasian equilibrium. In either case, a Walrasian equilibrium exists.■

7 Uniqueness, Stability, and Testability

We now provide an introduction to some of the most important positive properties of general equilibrium theory. We will ask when a Walrasian equilibrium is unique, whether it is stable in the sense that it can be reached by a simple price adjustment process, and we will look at whether Walrasian equilibrium imposes substantive restrictions on observable data.

This lecture will be less formal than previous lectures, mostly going through each of these topics at a rather high level. We have already alluded to the answers to some of these questions: no, Walrasian equilibria need not be unique, and no, it is not the case that a simple price adjustment process will always converge to a Walrasian equilibrium. We will first establish these results under general preferences. We will then focus on a special class of economies in which consumer preferences satisfy the *gross substitutes property*—when this property is satisfied, the model is particularly well-behaved: there will be a unique Walrasian equilibrium, and there will be a simple price-adjustment process that will always converge to it.

7.1 Uniqueness and Stability under Fairly General Preferences

Uniqueness We will first look at the question of whether there is a globally unique Walrasian equilibrium. Recall from the previous lecture the definition of the aggregate excess demand function $z(p) = \sum_{i \in \mathcal{I}} z_i(p)$, where $z_i(p) = x_i(p, p \cdot \omega_i) - \omega_i$.

Let us consider a two-commodity, two-consumer economy and normalize $p_2 = 1$. We argued informally last time that a Walrasian equilibrium exists by claiming that $z_1(p_1, 1)$ is continuous in p_1, $z_1(p_1, 1) > 0$ for p_1 small, and $z_1(p_1, 1) < 0$ for p_1 large. By the intermediate-value theorem, there exists a p_1^* such that $z_1(p_1^*, 1) = 0$ and therefore $(p_1^*, 1)$ is a Walrasian equilibrium price vector.

Is there any reason to think that there is only one p_1^* at which $z_1(p_1^*, 1) = 0$? Yes, if $z_1(p_1, 1)$ is everywhere downward-sloping, and in some sense, this is the natural case. It just says that there is less aggregate demand for commodity 1 when p_1 is higher, and we will show later that when the economy satisfies the gross substitutes condition, this will always be the case. But there certainly are situations where $z_1(p_1, 1)$ is not always downard-sloping. There is the somewhat pathological case in which commodity 1 is a Giffen good, so that $x_{1,i}(p, w)$ is increasing in p_1 even holding w fixed. Even if neither good is a Giffen good, however, $x_{1,i}(p, p \cdot \omega_1)$ may be increasing in p_1 because consumer i's wealth is increasing in p_1, so an upward-sloping region of $z_1(p_1, 1)$ is not particularly implausible.

In the first lecture, we discussed an example in the Edgeworth box in which the two consumers' offer curves intersected at three equilibrium points, and in the first problem set, you were asked to solve for the set of Walrasian equilibria in a numerical example for which there were three equilibria. Recall the example from the problem set. Consumers' preferences and endowments are:

$$\begin{aligned} u_1(x_{1,1}, x_{2,1}) &= \left(x_{1,1}^{-2} + \left(\frac{12}{37}\right)^3 x_{2,1}^{-2}\right)^{-1/2}, \qquad \omega_1 = (1, 0), \\ u_2(x_{1,2}, x_{2,2}) &= \left(\left(\frac{12}{37}\right)^3 x_{1,2}^{-2} + x_{2,2}^{-2}\right)^{-1/2}, \qquad \omega_2 = (0, 1). \end{aligned}$$

If we normalize $p_2 = 1$, consumers' Marshallian demands for commodity 1 are:

$$x_{1,1}((p_1, 1), p_1) = \frac{p_1}{p_1 + \frac{12}{37} p_1^{1/3}}, \quad x_{1,2}((p_1, 1), 1) = \frac{1}{p_1 + \frac{37}{12} p_1^{1/3}},$$

and the aggregate excess demand for commodity 1 is therefore

$$z_1(p_1, 1) = \frac{1}{p_1 + \frac{37}{12} p_1^{1/3}} - \frac{\frac{12}{37} p_1^{1/3}}{p_1 + \frac{12}{37} p_1^{1/3}}.$$

Figure 11 plots $z_1(p_1, 1)$ and shows that there are three solutions to $z_1(p_1, 1) = 0$. You might recall from the first problem set that for $p_1^* \in \left\{\frac{27}{64}, 1, \frac{64}{27}\right\}$, there is a Walrasian equilibrium with prices $(p_1^*, 1)$.

There are two additional general points that you can see illustrated in both Figures 5(a) and 11. The first is that if you were to perturb the economy slightly by changing consumers' preferences or endowments by a tiny amount, this would not affect the fact that there are three Walrasian equilibria.

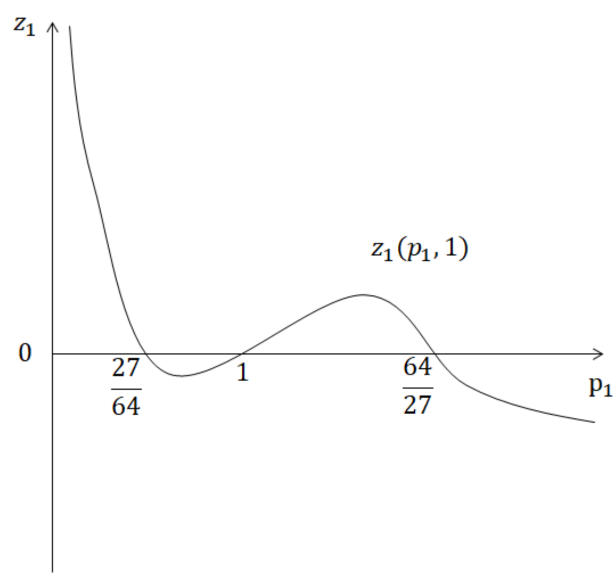


Figure 11: Multiple Walrasian equilibria

The second general point that these examples illustrate is that even though Walrasian equilibria may not be globally unique, they may be what is referred to as *locally unique* in the sense that there is no other Walrasian equilibrium price vector within a small enough range around the original equilibrium price vector. Figure 12 illustrates an example for which this is not the case. An equilibrium is not locally unique if its price vector p is the limit of a sequence of other equilibrium price vectors. This example shows that this can happen, but only if $z_1(p_1, 1)$ is flat and equal to zero over some interval of prices $[p_1^*, p_1^{**}]$. The important point to note about this example is that it is not *generic*: any small perturbation of $z_1(\cdot, 1)$

that would arise from, say, a change in endowments, would restore the property that there are a finite number of equilibria.

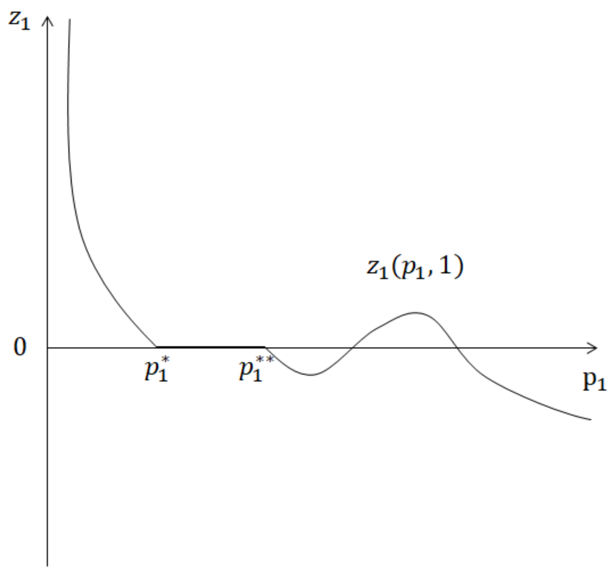


Figure 12: Walrasian Equilibria need not be locally unique

Exercise 11 (Adapted from MWG 17.D.1). Consider an exchange economy with two commodities and two consumers. Both consumers have homothetic preferences of the constant elasticity variety. Moreover, the elasticity of substitution is the same for both consumers and is small (i.e., commodities are close to perfect complements). Specifically,

$$u_1(x_{1,1}, x_{2,1}) = \left(2x_{1,1}^{\rho} + x_{2,1}^{\rho}\right)^{1/\rho} \text{ and } u_2(x_{1,2}, x_{2,2}) = \left(x_{1,2}^{\rho} + 2x_{2,2}^{\rho}\right)^{1/\rho},$$

and $\rho = -4$. The endowments are $\omega_1 = (1, 0)$ and $\omega_2 = (0, 1)$. Compute the excess demand function of this economy and find the set of competitive equilibria.

Tatonnement Stability One important aspect of Walrasian equilibrium that we have alluded to throughout the course but have not yet addressed is: where do Walrasian equilibrium prices come from? General equilibrium theory is quite weak on the kinds of price-adjustment processes that might lead to Walrasian equilibrium outcomes.

Walras proposed a process he called "tatonnement" whereby a fictitious Walrasian auctioneer gradually raises the price of commodities in excess demand and reduces the prices of those in excess supply until markets clear. This process is related to what the Walrasian auctioneer did in our proof of existence from last time, but not quite the same. In particular, the process last time adjusted prices discontinuously, but it was aimed at showing the existence of a fixed point for a particular operator, not at showing that the fixed point(s) of that operator could be found by iterating it from an arbitrary starting point.

Formally, consider the following continuous-time price-adjustment process $p(t)$:

$$\frac{dp(t)}{dt} = \alpha z(p(t)),$$

for some constant $\alpha > 0$. Given a starting price vector $p(0)$, the process raises prices for any commodities l for which $z_l(p(t)) > 0$ (i.e., for which there is excess demand), and it reduces prices for those for which $z_l(p(t)) < 0$.

The stationary points of this process are prices p at which $z(p) = 0$: Walrasian equilibrium prices. An equilibrium price vector p^* is said to be *locally stable* if the price-adjustment process converges to p^* from any "nearby" price vectors, and it is *globally stable* if the process converges to p^* from *any* initial starting price vector. Does this process converge to a Walrasian equilibrium price vector? When there are only two commodities, and the economy satisfies properties $(A1)$, $(A2')$, $(A3)$, and $(A4)$, this process does in fact converge, as Figure 13 highlights. Here, we can also see that p_1^* and p_1^{***} are locally stable, and p_1^{**} is not, and none of the equilibrium price vectors is globally stable.

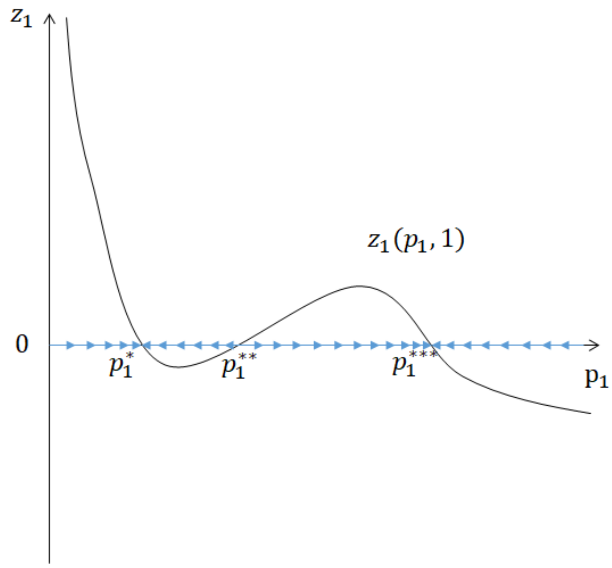


Figure 13: Tatonnement process for two commodities

This price-adjustment process gives us a way to study how equilibrium prices might be reached, but it has several drawbacks. First, the process itself is a conceptual exercise rather than a practical one—the GE model predicts that no one will trade at non-equilibrium prices. Second, if one were to try to implement this process by asking consumers how much they would demand at different price levels, then they would be unlikely to want to report their demands truthfully. Finally, the main drawback with this procedure is that it does not in general converge to an equilibrium price vector. In a famous paper, Scarf (1960) provided several examples in which the process does not converge when there are more than two commodities. We will show in the next section, however, that there are classes of economies for which it does converge.

7.2 Uniqueness and Stability under Gross Substitutes

In this section, we will show that economies that satisfy the gross substitutes property have particularly nice properties: there is a unique Walrasian equilibrium (up to a normalization), it is globally stable, and it has nice comparative statics properties.

Recall from consumer theory that commodities k and l are gross substitutes if an increase in p_k increases the Marshallian demand for commodity l (and vice versa), holding wealth fixed. The analogous definition in general equilibrium is as follows.

Definition 4. *A Marshallian demand function $x(p, p \cdot \omega)$ satisfies* ***gross substitutes at endowment*** *ω if, for all prices p and p' with $p'_k > p_k$ and $p'_l = p_l$ for all $l \neq k$, we have $x_l(p', p' \cdot \omega) > x_l(p, p \cdot \omega)$ for all $l \neq k$.*

This definition of gross substitutes is more subtle than the definition you saw from consumer theory, since increasing p_k also increases the consumer's wealth. It is straightforward to show that if all commodities are gross substitutes in the consumer-theory sense and they are also normal goods (so that demand increases with wealth), then demand functions will satisfy the gross substitutes property for all possible (non-negative) endowments. It is not readily apparent from the definition of gross substitutes that the demand for commodity l is decreasing in p_l, but it is true: since demand is homogeneous of degree 0 in p, increasing p_l is the same as holding p_l fixed and decreasing all other prices. Since decreasing each of these other prices decreases demand for commodity l, so does decreasing all of them.

If each consumer i's demand function satisfies gross substitutes at ω_i, then so does aggregate demand $\sum_{i \in \mathcal{I}} x_i(p, p \cdot \omega_i)$. The property is restrictive, but it is satisfied by many common functional forms such as CES preferences: $u_i(x_i) = \left(\sum_{l \in \mathcal{L}} \alpha_l x_{l,i}^{\rho}\right)^{1/\rho}$ for $0 < \rho < 1$.

If aggregate demand satisfies the gross substitutes property, then there is a unique Walrasian equilibrium, as the following result shows.

Proposition 1. *If the aggregate excess demand function $z(\cdot)$ satisfies gross substitutes, the economy has at most one Walrasian equilibrium (up to a normalization).*

Proof of Proposition 1. We need to show that there is at most one (normalized) price vector p such that $z(p) = 0$. To see why this is the case, suppose $z(p) = z(p') = 0$ for two price vectors p and p' that are not collinear. By homogeneity of degree zero, we can normalize the price vectors in such a way that $p'_l \geq p_l$ for all $l \in \mathcal{L}$ and $p'_k = p_k$ for some commodity k. Then, to move from p to p', we can think about doing this in $L-1$ steps, increasing the prices of each commodity $l \neq k$ in turn. At each step where a component of the price vector increases strictly, the aggregate demand for commodity k must strictly increase, so that $z_k(p') > z_k(p) = 0$. Moreover, there must be at least one such k, since p is not collinear with p', yielding a contradiction.■

When aggregate demand satisfies the gross substitutes property, not only is there a unique Walrasian equilibrium, but the tatonnement price-adjustment process we described above globally converges to it. To establish this result, we will first prove a lemma.

Lemma 4. *Suppose that the aggregate excess demand function $z(\cdot)$ satisfies gross substitutes and that $z(p^*) = 0$. Then for any p not collinear with p^*, $p^* \cdot z(p) > 0$.*

Proof of Lemma 4. We will give the proof in the $L = 2$ case. Normalize $p_2 = p_2^* = 1$. Then

$$\begin{aligned} p^* \cdot z(p) &= (p^* - p) \cdot (z(p) - z(p^*)) \\ &= (p_1^* - p_1)(z_1(p) - z_1(p^*)) > 0. \end{aligned}$$

The first equality uses Walras's Law (giving us that $p \cdot z(p) = 0$) and the fact that p^* is a Walrasian equilibrium (so that $z(p^*) = 0$). The second equality uses the normalization $p_2 = p_2^* = 1$. The inequality follows from the gross substitutes property: $p_1 > p_1^*$ implies $z_1(p) < z_1(p^*)$ and $p_1 < p_1^*$ implies $z_1(p) > z_1(p^*)$.■

With this Lemma, we can prove that the tatonnement process converges to the unique (up to normalization) Walrasian equilibrium price vector p^*.

Proposition 2. *Suppose that the aggregate excess demand function $z(\cdot)$ is satisfies the gross substitutes property and that p^* is a Walrasian equilibrium price vector. Then the price-adjustment process $p(t)$ defined by $dp(t)/dt = \alpha z(p(t))$, with $\alpha > 0$, converges to p^* for any initial condition $p(0)$.*

Proof of Proposition 2. To prove this result, we will show that the squared distance between $p(t)$ and p^* decreases monotonically in t. Let $D(p) = \frac{1}{2}\sum_{l\in\mathcal{L}}(p_l - p_l^*)^2$ denote the distance between p and p^*. Then

$$\begin{aligned}\frac{dD(p(t))}{dt} &= \sum_{l\in\mathcal{L}}(p_l(t) - p_l^*)\frac{dp_l(t)}{dt} \\ &= \alpha\sum_{l\in\mathcal{L}}(p_l(t) - p_l^*)z_l(p(t)) \\ &= -\alpha p^* \cdot z(p) \leq 0,\end{aligned}$$

where the third equality uses Walras's law. By the previous lemma, the last inequality is strict unless p is collinear with p^*. Since $D(p(t))$ is monotonic and bounded, it must converge to some value $\delta \geq 0$. If $\delta = 0$, we are done. If $\delta > 0$, then there is a contradiction, because continuity of aggregate demand implies that $p^* \cdot z(p(t))$ is bounded away from 0 for all $p(t)$ bounded away from p^*.■

Finally, economies with the gross substitutes property have nice comparative statics. Any change that raises excess demand for commodity k will increase its equilibrium price. As an example, suppose there are two commodities and normalize $p_2 = 1$. Suppose also that commodity 1 is a normal good for all consumers. Consider an increase in the aggregate endowment for commodity 2. For any price p_1, this will increase aggregate demand for commodity 1 and hence increase $z_1(\cdot, 1)$.

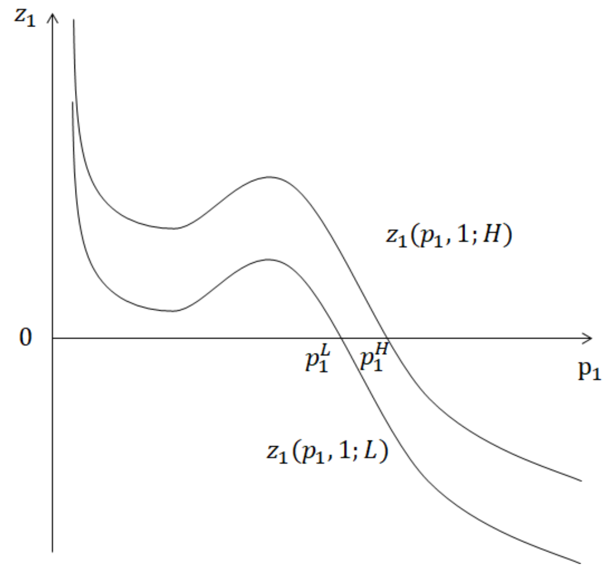


Figure 14: Comparative statics

Figure 14 compares the aggregate excess demand functions for two economies: one $(z_1(\cdot, 1; L))$ with a low aggregate endowment of commodity 2 and one $(z_1(\cdot, 1; H))$ with a high aggregate endowment of commodity 2. The curve $z_1(\cdot, 1; H)$ lies above $z_1(\cdot, 1; L)$ and because it is continuous and crosses zero once, the new equilibrium price vector must have a higher price for commodity 1.

7.3 Empirical Content of GE

As we just saw, whether there is a unique Walrasian equilibrium and whether Walrasian equilibria are stable depended critically on the structure of the economy's aggregate excess demand function $z(\cdot)$.

What do we know in general about the structure of aggregate excess demand? We proved that under assumptions $(A1)$, $(A2')$, $(A3)$, and $(A4)$ about consumer preferences and endowments that $z(\cdot)$ is continuous, homogeneous of degree zero in p, it satisfies Walras's Law, and $\lim_{p\to 0} z(p) \to \infty$. But, as Sonnenschein (1973) showed for the case of two commodities, and Mantel (1974) and Debreu (1974) showed more generally, the assumption of consumer maximization alone imposes no further restrictions on $z(\cdot)$. This is a very negative result, since it implies that even if we observe an economy in a Walrasian equilibrium with price vector p, it is possible for the same economy to have an arbitrary number of Walrasian equilibria with arbitrary stability properties.

Theorem 5 (Sonnenschein-Mantel-Debreu Theorem). *For any closed and bounded set of positive prices $\mathcal{P} \subseteq \mathbb{R}^L_{++}$ and any function $f : \mathcal{P} \to \mathbb{R}^L$ satisfying continuity, homogeneity of degree* 0*, and Walras's Law, there exists an exchange economy with L consumers with continuous, strictly convex, and monotone preferences whose aggregate excess demand function coincides with f on $\mathcal{P}$.*

We omit the proof here. See MWG Chapter 17.E for a proof in the $L = 2$ case and a discussion about the more general proof. Roughly speaking, the structure of the proof begins with a candidate excess demand function $f(p)$ that is continuous, homogeneous of degree 0, and satisfies Walras's Law and reverse engineers a set of consumer preferences and endowments that generate $f(p)$ as the aggregate excess demand function. The ability to do so requires a lot of flexibility in specifying consumer preferences that feature potentially strong income effects as well as the ability to specify consumers' endowments. A common interpretation of this theorem is that "anything goes" in general equilibrium theory. That is, without making strong assumptions on preferences: (i) pretty much any comparative statics result could be obtained in a general equilibrium model, and (ii) general equilibrium theory has essentially no empirical content. This is not quite right, though, as we will now see.

Brown and Matzkin (1996) prove an important result showing that if an economist is able to observe endowments as well as prices, then the Walrasian model is in principle testable. That is, there are endowment and price pairs $\left(p, (\omega_i)_{i\in\mathcal{I}}\right)$ and $\left(p', (\omega_i')_{i\in\mathcal{I}}\right)$ such that if p is a Walrasian equilibrium price vector given a fixed set of consumers with endowments $(\omega_i)_{i\in\mathcal{I}}$, then if the same set of consumers instead had endowments $(\omega_i')_{i\in\mathcal{I}}$, p' could not be a Walrasian equilibrium price vector.

Theorem 6 (Brown-Matzkin Theorem). *There exists price-endowment pairs* $\left(p, (\omega_i)_{i \in \mathcal{I}}\right)$ *and* $\left(p', (\omega'_i)_{i \in \mathcal{I}}\right)$ *such that there do not exist monotone preferences* $(u_i)_{i \in \mathcal{I}}$ *such that* p *is a Walrasian equilibrium price vector for the exchange economy* $(u_i, \omega_i)_{i \in \mathcal{I}}$ *and* p' *is a Walrasian equilibrium price vector for the exchange economy* $(u_i, \omega'_i)_{i \in \mathcal{I}}$.

Proof of Theorem 6. We can prove this theorem in the case of two consumers and two commodities. Consider the two Edgeworth boxes in Figure 15. Because p is a Walrasian equilibrium price vector given endowment ω, consumer 1 must weakly prefer some bundle on the segment A to any bundle on the segment B. By monotonicity, for every point on the segment A', there is some point on B that consumer 1 strictly prefers. There is therefore some bundle on A that is preferred by consumer 1 to every bundle on A'. If p' is a Walrasian equilibrium price vector given ω', we have a contradiction: every bundle on A is available to consumer 1 at prices p', yet she chooses a bundle on A'.■

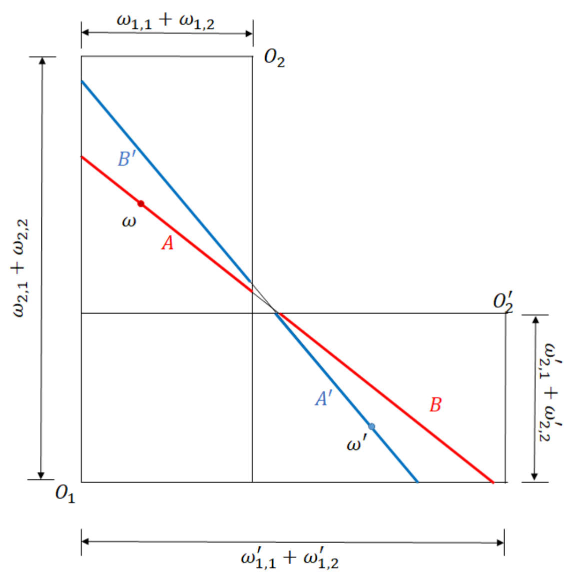


Figure 15: Brown-Matzkin theorem

The Brown-Matzkin theorem shows that, in order to construct the arbitrary excess demand functions that the proof of the Sonnenschein-Mantel-Debreu theorem requires, you really need the flexibility in specifying arbitrary endowments in addition to flexibility in specifying preferences. It also illustrates a more general point that, even if at its highest level, a theory imposes little structure on endogenous variables, imposing more structure on the theory typically imposes more structure on its implications.

8 Foundations of General Equilibrium Theory

When we talked about the normative properties of the GE model in the first week of class—in particular, the first welfare theorem—we were focused on questions about *what* is true in equilibrium. We did not really address the question of *why* it is that Walrasian equilibrium allocations are Pareto optimal, because in some sense, the framework itself is ill-equipped to answer this question. In order to get a better sense for why Walrasian equilibrium allocations are Pareto efficient, and in order to get a better sense for when the GE framework is an appopriate way of viewing the world, we will have to take a step outside the model in order to provide some foundations for the model itself.

We will ask two sets of questions. First, when might we expect Walrasian equilibrium allocations to arise? Second, when and why might we expect Pareto-optimal allocations to arise? To address these questions, we will consider two alternative foundations for Walrasian equilibrium, and in both, the answer to this question will be: when individuals in the economy are "small."

What is potentially problematic about Walrasian equilibrium as a description of the economy is that prices are endogenous variables, but they are not the explicit choices of anyone in the economy. In reality, individuals set prices—they bid in auctions, they post prices in their stores, they negotiate prices with their suppliers. Setting prices are individual decisions. One of the main premises of GE is that, when individuals are small relative to the economy, "market forces" pin down the prices at which trade occurs, and although it may be possible, it would be unwise for individuals to choose any other prices or any other consumption bundles. The question is: what are "market forces?"

Providing microfoundations for GE theory boils down to providing an answer to the question, "Under what conditions do small individuals lack market power, in the sense that they are forced to trade only at competitive prices?" There are two main approaches we will consider here. The first is the *cooperative game theory approach* in which the primitives of the model remain the agents' endowments and preferences, and the process of price-setting and trade is still specified only implicitly. Under this approach, however, the solution concept we will be using does not directly involve prices. Yet as the economy becomes large, consumers will receive the same allocations they would receive in a Walrasian equilibrium.

The second approach we will consider is the *non-cooperative game theory approach* in which we will explicitly model price-setting and trade and think about the (Nash) equilibria of the resulting trading processes. Consumers will have actions that can directly affect the prices they and other consumers pay for different commodities, and therefore equilibria will generically be inefficient. In the limit as the economy becomes large, however, consumers' actions will have little effect on prices, and equilibrium consumption choices will converge to Walrasian equilibrium allocations.

A benefit of the non-cooperative approach relative to the cooperative approach is that Pareto optimality will arise as a result rather than as a maintained assumption. We will therefore be able to develop some deeper intuition for why, exactly, Walrasian equilibrium allocations are Pareto optimal. We will then conclude this section with a brief discussion of who gets what in equilibrium and how under the notion of *competitive equilibrium* in which consumers' impact on prices is miniscule, consumers receive exactly what they contribute to the economy. A common theme in these approaches is that while Walrasian equilibrium is not necessarily a good description of small-numbers interactions, it may be a reasonable description of large-numbers interactions.

8.1 The Cooperative Approach

Going back to his 1881 classic, *Mathematical Psychics*, Edgeworth proposed that in economies with a small number of individuals, the outcome might be "indeterminate." We saw an example of this in the first week when we looked at Edgeworth boxes—with two consumers, Edgeworth believed that the only prediction we could reasonably make is that the final allocation would lie on the contract curve: the set of Pareto optimal allocations that are preferred by each consumer to her endowment. But he also conjectured that as the number of consumers grows, the scope for contracting among different consumers grows, and the resulting contract curve shrinks until it reaches only the set of Walrasian equilibrium allocations.

In the 20th century, economists formalized a version of this argument in what is known as the *core convergence theorem*. In order to describe what the core convergence theorem is, we will first have to define what the *core* is. The idea of the core is that it is the set of allocations for which no group of consumers can get together and trade with each other and do strictly better. Formally, consider a pure exchange economy $\mathcal{E}$ with I consumers whose preferences are continuous, strictly convex, and strongly monotone. We will define the core by defining what it is not. We will say that a coalition $\mathcal{S} \subseteq \mathcal{I}$ of consumers *blocks* an allocation if its members can all do strictly better by trading among themselves. In the case of $\mathcal{I} = \{1, 2\}$ that we considered in the Edgeworth box, any allocation that is not in the Pareto set is blocked by the coalition $\{1, 2\}$, and any allocation in the Pareto set but not on the contract curve is blocked either by coalition $\{1\}$ or by coalition $\{2\}$.

Definition 5. *A coalition* $\mathcal{S} \subseteq \mathcal{I}$ **blocks** *the allocation* $x^* = (x_1^*, \ldots, x_I^*) \in \mathbb{R}_+^{LI}$ *if there exists another allocation such that*:

1. $u_i(x_i) > u_i(x_i^*)$ *for all* $i \in \mathcal{S}$, *and*

2. $\sum_{i \in \mathcal{S}} x_i \leq \sum_{i \in \mathcal{S}} \omega_i$.

The core is the set of feasible unblocked allocations.

Definition 6. *A feasible allocation x^* is in the* **core** *if it is not blocked by any coalition. The core is therefore the set of unblocked feasible allocations.*

In terms of the Edgeworth box example, the core corresponds to the contract curve, since all other allocations are blocked by some coalition. The core convergence theorem provides conditions under which, when the economy grows large, the set of *core* allocations coincides with the set of *Walrasian equilibrium* allocations. We will break this claim up into two parts. First, we will show that any Walrasian equilibrium allocation is in the core. Then, we will show that any allocation that remains in the core as the economy grows large is a Walrasian equilibrium allocation.

Proposition 3. *Any Walrasian equilibrium allocation is in the core.*

Proof of Proposition 3. Let $\left(p^*, (x_i^*)_{i\in\mathcal{I}}\right)$ be a Walrasian equilibrium. Suppose $(x_i^*)_{i\in\mathcal{I}}$ is not in the core. Then there is some coalition $\mathcal{S} \subseteq \mathcal{I}$ that can block x^* with some other feasible allocation $\hat{x}$. Then $p^* \cdot \hat{x}_i > p^* \cdot \omega_i$ for all $i \in \mathcal{S}$ by consumer optimality. Since this holds for all $i \in \mathcal{S}$, we must also have $p^* \cdot \left(\sum_{i\in\mathcal{S}} \hat{x}_i\right) > p^* \cdot \left(\sum_{i\in\mathcal{S}} \omega_i\right)$. Since $p^* \geq 0$, this implies that $\sum_{i\in\mathcal{S}} \hat{x}_{l,i} > \sum_{i\in\mathcal{S}} \omega_{l,i}$ for some commodity l. But this means that $\hat{x}$ was not feasible to begin with.■

In order to establish the other direction of the core convergence theorem, we will have to define formally what we mean when we say that an economy grows large. As we know from the Edgeworth box example, when there are only two consumers, not every core allocation is a Walrasian equilibrium allocation. Whether this result remains true as we add more consumers to the economy depends on *how* we add more consumers to the economy. For example, if consumers 1 and 2 only care about their consumption of pens and pencils, and they are endowed with pens and pencils and nothing else, then if we add a bunch of other consumers who care only about their consumption of paper clips and have an endowment of paper clips and nothing else, then this will not do anything to make the terms of trade between consumers 1 and 2 more competitive.

If instead, we "grow the economy" by adding more consumers like consumer 1 (i.e., consumers who have the same preferences and endowment as consumer 1) and adding more consumers like consumer 2, then core allocations do begin to look more like Walrasian equilibrium allocations. Roughly speaking, the reason why is that if any particular consumer is getting a "good deal" from the rest of the consumers at a particular allocation, then the other consumers would prefer to cut her out of the deal and redistribute her net trade among themselves. This may not work when there are only a couple consumers in the economy because the excluded consumer may be hard to replace.

In order to formalize this argument, suppose there are H **types** of consumers $h \in \mathcal{H} = \{1, \ldots, H\}$. A type-$h$ consumer has preferences u_h and endowment ω_h. For each integer $N > 0$, we consider the N**-replica economy**, which is a pure exchange economy consisting of $I_N \equiv N \cdot H$ consumers, N of each type. We will refer to an allocation in which consumers of the same type consume the same consumption bundle as an **equal-treatment allocation**. The next lemma establishes that any core allocation of the N-replica economy is an equal-treatment allocation. Denote by $x_{h,n}$ the allocation of the n^{th} consumer of type h.

Lemma 5. *Suppose x is in the core of the N-replica economy, for some $N > 0$. Then all consumers of the same type receive the same allocation: $x_{h,n} = x_{h,m}$ for all $n, m \leq N$ and all types $h \in \mathcal{H}$.*

Proof of Lemma 5. We will proceed by way of contradiction. Suppose x is in the core of the N-replica economy for some $N > 0$, but for some type of consumer—without loss of generality, say type 1—not all consumers of that type receive the same allocation. We will want to show that in fact, such an allocation is not in the core. In particular, we will show that the coalition consisting of the worst-off consumer of every type can block the allocation x.

To see why this is true, let $\hat{x}_h = \frac{1}{N} \sum_n x_{h,n}$ denote the average allocation of type-h consumers. Without loss of generality, suppose that it is consumer number 1 of each type h

who is worst off within type h. By strict convexity of preferences, $u_h(\hat{x}_h) \geq u_h(x_{h,1})$ for all h, and $u_1(\hat{x}_1) > u_1(x_{1,1})$. The coalition $\{(1,1), \ldots, (H,1)\}$ can attain consumption vector $(\hat{x}_1, \ldots, \hat{x}_H)$ for its members, since feasibility implies

$$\sum_{h \in \mathcal{H}} \hat{x}_h = \frac{1}{N} \sum_{h \in \mathcal{H}} \sum_{n=1}^{N} x_{h,n} \leq \frac{1}{N} \sum_{h \in \mathcal{H}} \sum_{n=1}^{N} \omega_h = \sum_{h \in \mathcal{H}} \omega_h.$$

Finally, continuity and strong monotonicity of preferences imply that the consumption vector $(\hat{x}_1, \ldots, \hat{x}_H)$ can be perturbed to satisfy $u_h(\hat{x}_h) > u_h(x_{h,1})$ for all h, so the strict inequalities required to apply the definition of blocking are satisfied.■

This Lemma shows that any core allocation for an N-replica economy takes the form of a **type allocation** $(x_1, \ldots, x_H) \in \mathbb{R}_+^{LH}$, where each consumer of type h receives allocation x_h. Let $\mathcal{C}_N \subseteq \mathbb{R}_+^{LH}$ be the set of core allocations in the N-replica economy. Note that the set of core allocations shrinks as we replicate the economy: $\mathcal{C}_{N+1} \subseteq \mathcal{C}_N$ for all N. This is because any type allocation that is blocked by some coalition in the N-replica economy will be blocked by exactly the same coalition in the $N+1$-replica economy. At the same time, from Proposition 3, we know that the the set of Walrasian equilibrium allocations is independent of N and is always contained in $\mathcal{C}_N$. Debreu and Scarf (1963) proved that as $N \to \infty$, the set $\mathcal{C}_N$ shrinks to exactly the set of Walrasian equilibrium allocations. The version of the theorem we will prove will rely on two additional assumptions about preferences and endowments, although these assumptions can be relaxed.

Assumption A1' (continuous differentiability). For all consumers of type $h \in \mathcal{H}$, u_h is continuously differentiable.

Assumption A4' (interiority). For each $h \in \mathcal{H}$, ω_h is strictly preferred to any consumption bundle x_h that is not strictly positive.

Theorem 7 (Core Convergence Theorem). *Suppose $\mathcal{E}$ satisfies $(A1'), (A2'), (A3), (A4')$. If $x \in \mathcal{C}_N$ for all N, then x is a Walrasian equilibrium allocation.*

Proof of Theorem 7. At a high level, the proof of this theorem first argues that if $x \in \mathcal{C}_N$

for all N, then it is Pareto-optimal, which means that marginal rates of substitutions are equal across consumers and proportional to a price vector that will be used to construct a Walrasian equilibrium. It then argues that if at this price vector, some type of consumer is getting a "good deal" in that they consuming a bundle that is more expensive than their endowment, then $N-1$ consumers of this type along with all the other consumers in the economy can form a blocking coalition. This means that no type of consumer can be getting a good deal if $x \in \mathcal{C}_N$ for all N. The proof concludes with an argument that if no consumers are getting a good deal at $x \in \mathcal{C}_N$ for all N, then x can be decentralized as a Walrasian equilibrium allocation.

Step 1. Pareto-optimal allocations equate marginal rates of substitution across consumers and can be used to construct candidate prices.

Take an $x^* \in \mathcal{C}_N$ for all N. Since x^* is in the core, it is a Pareto-optimal allocation. Assumptions $(A1')$ and $(A4')$ ensure that at x^*,

$$\frac{\partial u_h/\partial x_{l,h}}{\partial u_h/\partial x_{l',h}} = \frac{\partial u_{h'}/\partial x_{l,h'}}{\partial u_{h'}/\partial x_{l',h'}} \text{ for all } h, h', l, l'.$$

Construct a price vector p^* for which $p_1^* = 1$, and

$$p_l^* = \frac{\partial u_h/\partial x_{l,h}}{\partial u_h/\partial x_{1,h}} \text{ for any } h,$$

so that relative prices match relative marginal utilities. We will now argue that (p^*, x^*) is a Walrasian equilibrium.

Step 2. No consumer types are getting a "good deal" at p^*.

Suppose that type-1 consumers are getting a "good deal" in the sense that their consumption is worth more than their endowment at prices p^*: $p^* \cdot x_1^* > p^* \cdot \omega_1$. We want to show that if this is the case, then x^* is not, in fact, in $\mathcal{C}_N$ for all N. To see why this is the case, note the marginal utility to any consumer type h of consuming an additional ε amount of consumer 1's net trade, $x_1^* - \omega_1$, is, to first order,

$$\varepsilon \sum_{l \in \mathcal{L}} \frac{\partial u_h}{\partial x_{l,h}} \left(x_{l,1} - \omega_{l,1}\right).$$

Since $p^* \cdot (x_1^* - \omega_1) > 0$, and the vector $(\partial u_h / \partial x_{1,h}, \ldots, \partial u_h / \partial x_{L,h})$ is proportional to p^*, this marginal utility is strictly positive. For $\varepsilon > 0$ sufficiently small, therefore, each consumer type h strictly prefers consuming $x_h^* + \varepsilon \left(x_1^* - \omega_1\right)$ to consuming x_h^*.

Now, consider allocation x^* in the N-replica economy. Suppose the coalition $\mathcal{S}$ consisting of everyone except a single type-1 consumer proposes an allocation that gives each coalition member of type h consumption $\hat{x}_h = x_h^* + \frac{1}{NH-1}\left(x_1^* - \omega_1\right)$. This allocation $\hat{x}$ is feasible for the coalition (you can check MWG, p. 658 for the argument for feasibility). Moreover, by the argument in the previous paragraph, if N is sufficiently large, $\hat{x}$ is strictly preferred to x^* by every coalition member. The coalition $\mathcal{S}$ therefore blocks the allocation x^*, so x^* is not in $\mathcal{C}_N$ for N sufficiently large. This contradicts the hypothesis that $p^* \cdot x_1^* > p^* \cdot \omega_1$, so it must be the case that no consumer types are getting a "good deal."

Step 3: Show that (p^*, x^*) is a Walrasian equilibrium.

From the previous step, we know that x_h^* is affordable for type h at prices p^* for all types h: $p^* \cdot x_h^* \leq p^* \cdot \omega_h$ for all $h \in \mathcal{H}$. The bundle x_h^* also satisfies consumer optimality. This is because under our interiority, differentiability, and convexity assumptions, each consumer type h will choose a consumption bundle that equates $\frac{\partial u_h}{\partial x_{l,h}} / p_l^*$ across commodities and therefore will optimally choose x_h^* at prices p^*.

Finally, since x^* is Pareto-optimal, it must also be feasible: $N \sum_{h \in \mathcal{H}} x_h^* \leq N \sum_{h \in \mathcal{H}} \omega_h$. Since preferences are monotone, this inequality must hold with equality, so the market-clearing condition is also satisfied. The vector (p^*, x^*) is therefore a Walrasian equilibrium.■

The core convergence theorem is an important result that is probably the best-known statement of the idea that large markets are approximately competitive. Note that there are no prices in the notion of the core. Yet what the core convergence theorem is saying is that in a sufficiently large economy, any allocation in the core corresponds to exactly what consumers would consume at equilibrium prices in a Walrasian equilibrium. Now, where do we see "competition" or "market forces" in the core convergence theorem? We see it in the second step of the proof, which is the only step that used the result that $x \in \mathcal{C}_N$ for all N. Competition ensures no consumers are getting an especially good deal from everyone else—if they were, then the rest of the economy could, in some sense, cut them out and make themselves better off.

The theorem itself has a number of shortcomings, however. First, the notion of a replica economy is extreme. We typically think of each individual as being unique, yet the thought experiment the core convergence theorem carries out requires that there are, in the limit, infinitely many people who have exactly the same preferences and endowments as you. Second, the theorem itself is not an approximation result—it does not say that for any finite N, any allocation in the core is *approximately* a Walrasian equilibrium allocation, since it does not say anything about distance. There is a large literature at the intersection of cooperative game theory and general equilibrium theory that tries to extend this result into something that is more convincing. One branch (following Arrow and Hahn, 1971, and others) relaxes the assumption of exact replication and tries to say something about core allocations in large but finite economies. Another branch (following Aumann, 1964) instead looks directly at economies with a continuum of consumers, for which the core convergence theorem provides an exact equivalence between core allocations and Walrasian equilibrium allocations.

8.2 The Non-Cooperative Approach

The cooperative approach imposes no structure on the underlying trading institutions and as a result, it has little to say about how prices are determined and under what conditions they are likely to correspond to Walrasian equilibrium prices. In contrast, under the non-cooperative approach, individual consumers make decisions that "aggregate up" to determine prices.

Suppose there are I consumers, a set $\mathcal{P} \subseteq \mathbb{R}^L$ of possible price vectors, and a set $\mathcal{A}$ of **market actions**. Each consumer $i \in \mathcal{I}$ has a set $\mathcal{A}_i \subset \mathcal{A}$ and an endowment vector $\omega_i \in \mathbb{R}^L$. For each $a_i \in \mathcal{A}_i$ and $p \in \mathcal{P}$, a **trading rule** assigns a net trade vector $g(a_i; p) \in \mathbb{R}^L$ to consumer i, satisfying $p \cdot g(a_i; p) = 0$. Given a vector of market actions $a = (a_1, \ldots, a_I)$, a market-clearing process generates a price vector $p(a) \in \mathcal{P}$. Throughout, we will assume each i has a utility function of the form $u_i(g(a_i; p) + \omega_i)$. An equilibrium of the resulting game is just a Nash equilibrium.

Definition 7. *The profile $a^* = (a_1^*, \ldots, a_I^*)$ of market actions is a* ***trading equilibrium*** *if, for every consumer $i \in \mathcal{I}$,*

$$u_i(g(a_i^*; p(a^*)) + \omega_i) \geq u_i\left(g\left(a_i; p\left(a_i, a_{-i}^*\right)\right) + \omega_i\right) \text{ for all } a_i \in \mathcal{A}_i.$$

We will consider a particular trading rule referred to as Shapley and Shubik's (1977) **trading posts**. It is not particularly realistic, but it does form a complete general equilibrium model in which all consumers interact strategically. Suppose there are I consumers and L commodities. Commodity L, which we will call "money," is treated differently from the other commodities, and we normalize its price to 1. For each of the other $L-1$ commodities, there is a *trading post* at which consumers can exchange money for the commodity.

At each trading post $l \leq L-1$, each consumer i places bids $a_{l,i} = \left(a'_{l,i}, a''_{l,i}\right) \in \mathbb{R}^2_+$. The first value, $a'_{l,i}$, is interpreted as the amount of commodity l that consumer i is willing to put up for sale in exchange for money. The second value, $a''_{l,i}$ is the amount of money that she puts up in exchange for commodity l. These bids must therefore satisfy $a'_{l,i} \leq \omega_{l,i}$, and $\sum_{l \leq L-1} a''_{l,i} \leq \omega_{L,i}$. Given the consumers' bids, the price of commodity l is set to be equal to the total amount spent on commodity l divided by the total quantity of commodity l supplied:

$$p_l = \frac{\sum_{i \in \mathcal{I}} a''_{l,i}}{\sum_{i \in \mathcal{I}} a'_{l,i}}.$$

Each consumer i receives allocation $x_{l,i} = g_l(a_i; p) + \omega_{l,i}$, where

$$g_l(a_i; p) = \frac{a''_{l,i}}{p_l} - a'_{l,i}$$

for all $l \leq L-1$ and $x_{L,i} = \omega_L - \sum_{l=1}^{L-1} a''_{l,i}$.

If there is a large number of consumers trading each commodity, then each consumer's bids would have a negligible effect on prices, and each consumer's allocation will be arbitrarily close to the solution to their problem

$$\max_{x_i \in \mathbb{R}^L_+} u_i(x_i) \text{ s.t. } p \cdot x_i \leq p \cdot \omega_i.$$

Thus, even though prices *are* determined as the aggregation of individual consumers' actions, when the economy is sufficiently large, each *individual* consumer's actions have no effect on prices. Under this approach, price-taking behavior is therefore a result rather than an assumption. We will refer to the resulting equilibrium as a *competitive equilibrium*.

One important difference between the cooperative approach and the non-cooperative approach to the foundations of GE theory is that under the cooperative approach, the allocations we considered were always Pareto optimal. In contrast, under the non-cooperative approach, allocations are *not* Pareto optimal for any finite market size. This is true for the same reason that a monopolist supplies too little in equilibrium: at the margin, each consumer's decisions affect other consumers because they affect prices. In small economies, what are known as *pecuniary externalities*—externalities that arise through prices themselves—matter for economic efficiency. When the economy is large, under this approach, there is a Nash equilibrium in which consumers effectively take prices as given because their individual actions have a negligible impact on prices. As a result, there are no pecuniary externalities, and the resulting allocation is a Walrasian equilibrium allocation and is therefore Pareto optimal. Price-taking, under this approach, is a result of large numbers, not an assumption. And Pareto optimality is a result of large numbers, not an assumption. Only the non-cooperative approach can, therefore, really tell us anything about *why* Walrasian equilibrium allocations are Pareto optimal.

8.3 Who Gets What? The No-Surplus Condition

This section concludes our discussion of the competitive foundations of general equilibrium theory. In particular, we will put a little bit more structure on the claim I made at the end of the previous section that there are no pecuniary externalities in large markets. We will ask whether Walrasian equilibria can be characterized by the idea that consumers get exactly what they contribute to the welfare of society. To answer this question, we will consider a special class of preferences in which the notion of the *welfare of society* is well-defined. In particular, suppose there are H types of consumers, $h \in \mathcal{H} = \{1, \ldots, H\}$, and each type of consumer is endowed with ω_h and has *quasi-linear preferences.*

Assumption QL (quasilinearity). For each type $h \in \mathcal{H}$, there is a concave, differentiable, strictly increasing function $v_h(x_{1,h}, \ldots, x_{L-1,h})$ such that type h preferences are $u_h(x_h) = v_h(x_{1,h}, \ldots, x_{L-1,h}) + x_{L,h}$, where $x_h \in \mathbb{R}_+^{L-1} \times \mathbb{R}$.

When consumers have quasilinear preferences, commodity L is what is referred to as the **money commodity**. It is a commodity for which all consumers have the same marginal utility and which consumers can consume any (positive or negative) amount of. The assumption of quasilinear preferences allows for cardinal measures of individuals' private rewards and their contribution to social welfare.

An economy is defined by a profile $(I_1, \ldots, I_H)$ of consumers of the different types, for a total of $I = \sum_{h\in\mathcal{H}} I_h$ consumers. For any economy, we can define the **social welfare**, $V(I_1, \ldots, I_H)$, as the solution to the following problem:

$$V(I_1, \ldots, I_H) = \max_{(x_h)_{h\in\mathcal{H}}} \sum_{h\in\mathcal{H}} I_h u_h(x_h)$$

subject to feasibility: $\sum_{h\in\mathcal{H}} I_h x_h \leq \sum_{h\in\mathcal{H}} I_h \omega_h$ and $x_{l,h} \geq 0$ for all $l \in \{1, \ldots, L-1\}$ and for all h. This function is homogeneous of degree one in its arguments, so we can describe the economy in terms of its per-capita social welfare $V(I_1/I, \ldots, I_H/I) = V(I_1, \ldots, I_H)/I$, and therefore if we extend the model to one in which there are a continuum of consumers, with mass $\mu_h \geq 0$ of type $h \in \mathcal{H}$ with $\sum_{h\in\mathcal{H}} \mu_h = 1$, we can write $\mu = (\mu_1, \ldots, \mu_H)$ and

$$V(\mu) = \max_{(x_h)_{h\in\mathcal{H}}} \sum_{h\in\mathcal{H}} \mu_h u_h(x_h) \tag{1}$$

subject to feasibility: $\sum_{h\in\mathcal{H}} \mu_h x_h \leq \sum_{h\in\mathcal{H}} \mu_h \omega_h$ and $x_{l,h} \geq 0$ for all $l \in \{1, \ldots, L-1\}$.

Given a continuum population of consumers, we can define a consumer of type h's **marginal contribution to social welfare** as $\partial V(\mu)/\partial \mu_h$. We will say that a feasible allocation $(x_h^*)_{h\in\mathcal{H}}$ is a **no-surplus allocation** if

$$u_h(x_h^*) = \frac{\partial V(\mu)}{\partial \mu_h} \text{ for all } h \in \mathcal{H}.$$

In other words, at a no-surplus allocation, each consumer is receiving in utility exactly what she contributes to social welfare. With this definition in mind, we can state the no-surplus characterization of Walrasian equilibrium.

Theorem 8 (No-Surplus Characterization). *For any continuum population $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_H) >> 0$, a feasible allocation $(x_1^*, \ldots, x_H^*) >> 0$ is a no-surplus allocation if and only if it is a Walrasian equilibrium allocation.*

Proof of Theorem 8. The structure of the proof is as follows. We will show that if $(x_h^*)_{h\in\mathcal{H}}$ is a no-surplus allocation, then it solves (1). We will then show that if $(x_h^*)_{h\in\mathcal{H}}$ solves (1), then $(x_h^*)_{h\in\mathcal{H}}$ is a Walrasian equilibrium allocation for a suitable price vector p^*. Finally, we will show that if $(x_h^*)_{h\in\mathcal{H}}$ is a Walrasian equilibrium allocation, then it is a no-surplus allocation.

Step 1: $(x_h^*)_{h\in\mathcal{H}}$ is no-surplus $\Rightarrow$ $(x_h^*)_{h\in\mathcal{H}}$ solves (1).

Suppose $(x_h^*)_{h\in\mathcal{H}}$ is a no-surplus allocation. We know that the function $V(\bar{\mu})$ is homogeneous of degree one in $\bar{\mu}$, so by Euler's formula, we can write

$$V(\bar{\mu}) = \sum_{h \in \mathcal{H}} \bar{\mu}_h \frac{\partial V(\bar{\mu})}{\partial \mu_h} = \sum_{h \in \mathcal{H}} \bar{\mu}_h u_h(x_h^*),$$

where the last equality used the fact that $(x_h^*)_{h \in \mathcal{H}}$ is a no-surplus allocation. This implies that $(x_h^*)_{h \in \mathcal{H}}$ is a solution to the social welfare-maximization problem for $\mu = \bar{\mu}$.

Step 2: $(x_h^*)_{h \in \mathcal{H}}$ solves (1) $\Rightarrow$ $(x_h^*)_{h \in \mathcal{H}}$ is a WE allocation.

Suppose now that $(x_h^*)_{h \in \mathcal{H}}$ is a feasible allocation that yields social welfare $V(\bar{\mu})$. Denote by p_l^*, $l = 1, \ldots, L$, the values of the Lagrange multipliers for commodity-l feasibility constraint, $\sum_{h \in \mathcal{H}} \bar{\mu}_h (x_{l,h} - \omega_{l,h}) \leq 0$, in the social-welfare-maximization problem. Because $u_h(\cdot)$ is quasilinear for all $h \in \mathcal{H}$, we will have $p_L^* = 1$ and $p_l^* = \partial u_h(x_h^*)/\partial x_{l,h}$ for all $l \in \{1, \ldots, L-1\}$ and for all $h \in \mathcal{H}$. It follows then that if we let $p^* = (p_1^*, \ldots, p_L^*)$, then $\left(p^*, (x_h^*)_{h \in \mathcal{H}}\right)$ is a Walrasian equilibrium.

Step 3: $(x_h^*)_{h \in \mathcal{H}}$ is a WE allocation $\Rightarrow$ $(x_h^*)_{h \in \mathcal{H}}$ is no-surplus.

Finally, we can apply the envelope theorem to (1) to get

$$\frac{\partial V(\bar{\mu})}{\partial \mu_h} = u_h(x_h^*) + p^* \cdot (\omega_h - x_h^*).$$

Since $(x_h^*)_{h \in \mathcal{H}}$ is consumer-optimal given prices p^*, by Walras's law, the second term is zero. We therefore have that $\partial V(\bar{\mu})/\partial \mu_h = u_h(x_h^*)$, so $(x_h^*)_{h \in \mathcal{H}}$ is a no-surplus allocation.■

Viewed in light of the no-surplus characterization of Walrasian equilibrium, we can finally develop some intuition for the first welfare theorem result that Walrasian equilibrium allocations are Pareto optimal. If, at the margin, each consumer is receiving exactly what she contributes to society's welfare, then in some sense, the rest of society is indifferent to her presence. Since each consumer is not affecting the welfare of the rest of society, of course each consumer doing the best she can—which she is, by the consumer optimality condition of Walrasian equilibrium—is going to lead to a result that is best for society.

It is important to realize that when there are a finite number of individuals in society, there generically do not exist any no-surplus allocations. The reason for this is that it is typically impossible to give each consumer the full extent of her marginal contribution while maintaining feasibility. For example, when there are only two consumers in the economy, each consumer's contribution to social welfare is equal to the utility she would get if she consumes her endowment plus the *entire gains from trade*, and we cannot simultaneously give both consumers the entire gains from trade. This means that in smaller economies, Walrasian equilibrium allocations generically are not no-surplus allocations.

9 Firms and Production in General Equilibrium

So far in this class, we have focused on pure exchange economies. In doing so, we have assumed that all the commodities in the economy come essentially from nowhere. In other words, we have completely abstracted away from the supply side of the economy. The GE framework can be readily extended to allow for firms and productions as long as two conditions are satisfied: (1) firms' production technologies do not exhibit increasing returns to scale, and (2) firms are price-takers. In this section, we will describe how to extend the GE framework to allow for production and we will show that versions of the welfare theorems and the existence theorem hold. We will then consider some simple examples and conclude with a result that shows that in this framework, we can think of the entire supply side of the economy as a single firm.

9.1 Extending the Framework

There are I consumers $i \in \mathcal{I}$ with utility functions $(u_i)_{i \in \mathcal{I}}$ defined over the consumption of L commodities $l \in \mathcal{L}$, and there are J firms $j \in \mathcal{J}$. Each firm possesses a production set $\mathcal{Y}_j \in \mathbb{R}^L$. The production set $\mathcal{Y}_j$ describes a set of feasible production plans: if $y_j = (y_{1,j}, \ldots, y_{L,j}) \in \mathcal{Y}_j$, then $y_{l,j} < 0$ means that commodity l is being used as an input, and

$y_{l,j} > 0$ means that commodity l is being produced as an output. The firms are owned by the households. **Consumer** i's **ownership share of firm** j is a $\theta_{i,j} \in [0,1]$. A **production economy** is then a collection $\mathcal{E} = \left(\left(u_i, \omega_i, (\theta_{i,j})_{j\in\mathcal{J}}\right)_{i\in\mathcal{I}}, (\mathcal{Y}_j)_{j\in\mathcal{J}}\right)$ of consumer preferences, consumer endowments, ownership shares, and production sets. Firm j takes prices $p \in \mathbb{R}^L$ as given and chooses a production plan $y_j \in \mathcal{Y}_j$ to maximize its profits:

$$\max_{y_j \in \mathcal{Y}_j} p \cdot y_j.$$

Our definition of Walrasian equilibrium extends naturally to production economies.

Definition 8. *A* ***Walrasian equilibrium*** *for the production economy* $\mathcal{E}$ *is a vector* $\left(p^*, (x_i^*)_{i\in\mathcal{I}}, \left(y_j^*\right)_{j\in\mathcal{J}}\right)$ *that satisfies:*

1. *Firm profit maximization: for all* $j \in \mathcal{J}$,

$$y_j^* \in \operatorname*{argmax}_{y_j \in \mathcal{Y}_j} p \cdot y_j,$$

2. *Consumer optimization: for all consumers* $i \in \mathcal{I}$,

$$x_i^* \in \operatorname*{argmax}_{x_i \in \mathcal{X}_i} u_i(x_i)$$

subject to

$$p \cdot x_i \leq p \cdot \omega_i + \sum_{j\in\mathcal{J}} \theta_{i,j} p \cdot y_j^*,$$

3. *Market-clearing: for all commodities* $l \in \mathcal{L}$

$$\sum_{i\in\mathcal{I}} x_{l,i}^* = \sum_{i\in\mathcal{I}} \omega_{l,i} + \sum_{j\in\mathcal{J}} y_{l,j}^*.$$

9.2 Assumptions on Production Sets

Just as we made a number of assumptions on consumer preferences and endowments, we will make several assumptions on production sets to ensure that a Walrasian equilibrium exists in a production economy. The simplest such assumption would be that Y_j is a convex and compact set for all firms $j \in \mathcal{J}$, but assuming that a production set is bounded is stronger than we need.

Assumption A5 (closed and convex): For all firms $j \in \mathcal{J}$, $\mathcal{Y}_j$ is closed and convex.

Assumption A6 (no production is feasible and free disposal): For all firms $j \in \mathcal{J}$, $0 \in \mathcal{Y}_j$, and for all $y_j \in \mathcal{Y}_j$, $\{y_j\} + \mathbb{R}^L_{-\,-} \subset \mathcal{Y}_j$.

These two assumptions rule out increasing returns to scale. To see why, note that if $y \in \mathcal{Y}_j$, then since $0 \in \mathcal{Y}_j$, so is αy_j for any $0 < \alpha < 1$, so it is always possible to scale down production or break it up into arbitrarily small productive units.

We will also need to make one further assumption on aggregate production to ensure that the supply side of the economy as a whole cannot produce something with nothing. We want to rule out, for example, situations where one firm can turn one pound of coffee beans into one cup of coffee, while another firm can turn one cup of coffee into two pounds of coffee beans. Define the **aggregate production set** to be the Minkowski sum of all the firms' production sets:

$$\mathcal{Y} = \sum_{j \in \mathcal{J}} \mathcal{Y}_j = \left\{ y : \text{there exist } y_1 \in \mathcal{Y}_1, \ldots, y_J \in \mathcal{Y}_J \text{ such that } y = \sum_{j \in \mathcal{J}} y_j \right\}.$$

The following assumption is sufficient to rule out the implausible situations described above.

Assumption A7 (irreversibility): $\mathcal{Y} \cap -\mathcal{Y} = \{0\}$.

It is worth spending some time thinking about why assumptions $(A6)$ and $(A7)$ rule out the situations I just described.

9.3 Welfare Theorems and Existence of Walrasian Equilibrium

The definitions of feasibility and Pareto efficiency are easily extended to production economies.

Definition 9. *An allocation and production plan* $\left((x_i)_{i\in\mathcal{I}}, (y_j)_{j\in\mathcal{J}}\right)$ *is* ***feasible*** *if*

$$\sum_{i\in\mathcal{I}} x_{l,i} \leq \sum_{i\in\mathcal{I}} \omega_{l,i} + \sum_{j\in\mathcal{J}} y_{l,j} \text{ for all } l \in \mathcal{L}.$$

A feasible allocation and production plan $\left((x_i)_{i\in\mathcal{I}}, (y_j)_{j\in\mathcal{J}}\right)$ is **Pareto optimal** if there is no other feasible allocation and production plan $\left((\hat{x}_i)_{i\in\mathcal{I}}, (\hat{y}_j)_{j\in\mathcal{J}}\right)$ satisfying $u_i(\hat{x}_i) \geq u_i(x_i)$ for all i, with strict inequality for at least one i'.

We can now state the extensions of the two welfare theorems.

Theorem 9 (First Welfare Theorem). *Suppose* $\left(p^*, (x_i^*)_{i\in\mathcal{I}}, (y_j^*)_{j\in\mathcal{J}}\right)$ *is a Walrasian equilibrium for production economy* $\mathcal{E}$. *Then if* (A2) *holds, the allocation and production* $\left((x_i^*)_{i\in\mathcal{I}}, (y_j^*)_{j\in\mathcal{J}}\right)$ *is Pareto optimal.*

The proof of the first welfare theorem for production economies is essentially the same as the proof for pure exchange economies. It is worth trying to extend each of the steps from our previous proof to allow for production. The second welfare theorem can be similarly extended.

Theorem 10 (Second Welfare Theorem). *Let* $\mathcal{E}$ *be a production economy that satisfies* (A1)−(A6). *Suppose* $\left((x_i)_{i\in\mathcal{I}}, (y_j)_{j\in\mathcal{J}}\right)$ *is Pareto optimal, and suppose* $x_i >> 0$ *for all* $i \in \mathcal{I}$. *Then there is a price vector* p*, ownership shares* $(\theta_{i,j})_{i\in\mathcal{I},j\in\mathcal{J}}$*, and endowments* $(\omega_i)_{i\in\mathcal{I}}$ *such that* $\left(p, (x_i)_{i\in\mathcal{I}}, (y_j)_{j\in\mathcal{J}}\right)$ *is a Walrasian equilibrium given these endowments and ownership shares.*

The proof of the second welfare theorem again relies on the separating hyperplane theorem. Whereas the separating hyperplane in the earlier proof separated the aggregate demand set (i.e., the set of points preferred to the endowment) from the endowment, the proof in production economies requires separation between the aggregate demand set and a suitably constructed aggregate supply set (i.e., the endowment plus the set of feasible aggregate production plans). Convexity of production sets is required in order to invoke the separating hyperplane theorem.

Finally, we can also show that if we impose all the assumptions $(A1)-(A7)$, then a Walrasian equilibrium exists.

Theorem 11 (Existence of Equilibrium). *Let $\mathcal{E}$ be a production economy that satisfies $(A1)-(A7)$. Then there exists a Walrasian equilibrium of $\mathcal{E}$.*

Exercise 12. Consider an economy with two consumers and two commodities. Consumer 1's endowment vector is $(\lambda, 0)$ and consumer 2's is $(\mu, 0)$. Each consumer's utility is the sum of their consumption of the two commodities. Consumer 1 owns a technology for transforming commodity 1 into commodity 2. The production function is $y_2 = y_1^2$ for $y_1 \leq 0$, where y_1 is the input of commodity 1.

(a) Does this economy have a Walrasian equilibrium?

(b) What allocation would a planner choose to maximize the sum of utilities? [Be careful about second-order conditions.]

(c) What is the core of this economy?

Exercise 13 (Adapted from MWG, 16.F.2-4). In the first week, we discussed the first-order conditions for Pareto optimality in exchange economies. This exercise asks you to extend these conditions to production economies with I consumers and J firms. Define the utility possibility set:

$$\mathcal{U} = \left\{(u_1, \ldots, u_I) \in \mathbb{R}^I : \exists \text{ feasible } (x_i)_{i\in\mathcal{I}}, (y_j)_{j\in\mathcal{J}} \text{ with } u_i(x_i) \geq u_i \text{ for all } i\right\}.$$

Assume the production set for firm j takes the form $\mathcal{Y}_j = \left\{y \in \mathbb{R}^L : F_j(y) \leq 0\right\}$, where $F_j(y) = 0$ defines firm j's **transformation frontier**, and $F_j : \mathbb{R}^L \to \mathbb{R}$ is twice continuously differentiable with $F_j(0) \leq 0$ and $\nabla F_j(y_j) >> 0$ for all $y_j \in \mathbb{R}^L$.

(a) Show that if F_j is a convex function, then $\mathcal{Y}_j$ is a convex set.

(b) [Optional] Show that if, for all $i \in \mathcal{I}$, $\mathcal{X}_i$ is convex and u_i is concave, and for all $j \in \mathcal{J}$, F_j is convex, then $\mathcal{U}$ is a convex set.

(c) Suppose $\lambda \geq 0$ is a non-zero vector of Pareto weights, and consider the Pareto problem

$$\max_{u\in\mathcal{U}} \lambda \cdot u.$$

Show that the optimality conditions for an interior solution (i.e. $x_i >> 0$ for all i) for this problem satisfy

$$\frac{\partial u_i/\partial x_{l,i}}{\partial u_i/\partial x_{l',i}} = \frac{\partial u_{i'}/\partial x_{l,i'}}{\partial u_{i'}/\partial x_{l',i'}} \text{ for all } i,i',l,l' \tag{1}$$

$$\frac{\partial F_j/\partial y_{l,j}}{\partial F_j/\partial y_{l',j}} = \frac{\partial F_{j'}/\partial y_{l,j'}}{\partial F_{j'}/\partial y_{l',j'}} \text{ for all } j,j',l,l' \tag{2}$$

$$\frac{\partial u_i/\partial x_{l,i}}{\partial u_i/\partial x_{l',i}} = \frac{\partial F_j/\partial y_{l,j}}{\partial F_j/\partial y_{l',j}} \text{ for all } i,j,l,l'. \tag{3}$$

(d) Consider the aggregate problem of maximizing the production of commodity 1 subject to minimum production levels $(\bar{y}_2, \ldots, \bar{y}_L)$ for the other commodities.

$$\max_{(y_1,\ldots,y_J)} \sum_{j \in \mathcal{J}} y_{1,j}$$

subject to

$$\sum_{j \in \mathcal{J}} y_{l,j} \geq \bar{y}_l \text{ for all } l = 2, \ldots, L$$

and

$$F_j(y_j) \leq 0 \text{ for all } j = 1, \ldots, J.$$

Show that the optimality conditions for this problem satisfy (2). What does these conditions imply about how production is carried out across firms in a Pareto optimal allocation?

9.4 A Constant Returns-to-Scale Example

For a production economy, we have to specify both consumers' preferences as well as firms' production sets. A simple class of production sets that satisfy assumptions ($A5$) – ($A7$) are linear production sets. Such production sets are convex cones spanned by finitely many rays.[6] There is a single firm that has access to M linear activities $a_m \in \mathbb{R}^L$, $a_m \in \mathcal{M} = \{a_1, \ldots, a_M\}$, and it can operate each activity at some level γ_m. Its production set $\mathcal{Y}$ is the convex hull of these activities:

$$\mathcal{Y} = \left\{ y \in \mathbb{R}^L : y = \sum_{m=1}^{M} \gamma_m a_m \text{ for some } \gamma \in \mathbb{R}_+^M \right\}.$$

[6] Let $\mathcal{X} \subset \mathbb{R}^N$ be a set that contains $\{0\}$. Take two vectors $x, y \in \mathcal{X}$. We will say that a vector $w = \alpha x + \beta y$, where $\alpha \geq 0, \beta \geq 0$ is a **conic combination** of the vectors x and y. If the set $\mathcal{X}$ contains all conic combinations of its elements, we say that $\mathcal{X}$ is a **convex cone**. The way to think about a convex cone is to imagine a convex set $\mathcal{A}$ located some distance from the origin. The convex cone generated by the set $\mathcal{A}$ is the set of all points that lie on a ray from the origin that goes through any point in $\mathcal{A}$. If $\mathcal{A}$ is a disk in $\mathbb{R}^2$, then the convex cone generated by $\mathcal{A}$ is what you would normally think of as a cone.

Assumption $(A5)$ is satisfied, and the free disposal part of assumption $(A6)$ is satisfied if the vectors

$$(-1,0,\ldots,0),(0,-1,0,\ldots,0),\ldots,(0,\ldots,0,-1)$$

are all in $\mathcal{M}$.

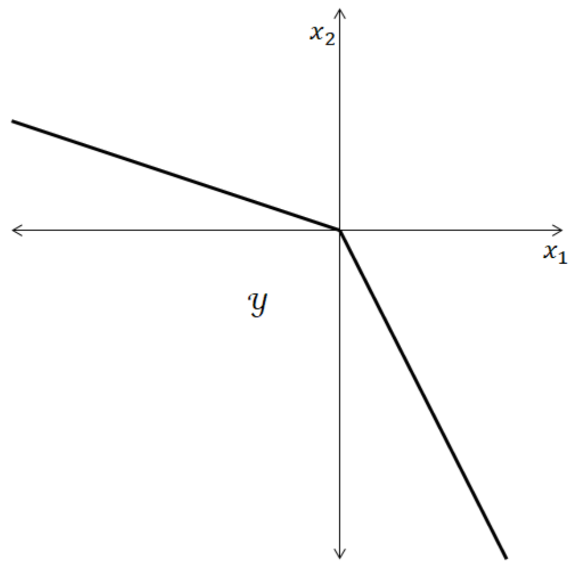


Figure 14: Linear Production Set

Figure 14 illustrates a linear production set in the special case of $M = 4$ and $L = 2$. There are two productive activities: activity 1 allows 2 units of commodity 2 to be converted into 1 unit of commodity 1. Activity 2 allows 3 units of commodity 1 to be converted into 1 unit of commodity 2. Activities 3 and 4 are the activities I described above that ensure that the free disposal assumption is satisfied. In this case,

$$\mathcal{M} = \{(1,-2),(-3,1),(0,-1),(-1,0)\}.$$

Note that the production set $\mathcal{Y}$ generated by $\mathcal{M}$ is the same production set that would be generated by activities $\{(1,-2),(-3,1)\}$ because $\frac{3}{5}(1,-2)+\frac{1}{5}(-3,1) = (0,-1)$ and $\frac{1}{5}(1,-2)+\frac{3}{5}(-3,1)=(-1,0)$.

Given a price vector p, a profit-maximizing plan exists if and only if $p \cdot a_m \leq 0$ for all $m = 1, \ldots, M$. If this were not the case, the firm's potential profits would be unbounded: if $p \cdot a_m > 0$ for some m, the firm could choose a sequence of production vectors $\gamma_m a_m$ with $\gamma_m \to \infty$, and its profits would increase without bound along that sequence. If $p \cdot a_m < 0$ for some m, then it is clear that optimal production $y = \sum_{m \in \mathcal{M}} \gamma_m a_m$ satisfies $\gamma_m = 0$.

When production sets are convex cones, as in this example, market clearing implies that equilibrium prices are pinned down by zero-profit conditions. This specification of production sets is not as much of a special case as it first appears—it is satisfied by any constant returns to scale production technology, including the Cobb-Douglas production functions you have used in macroeconomics.

9.5 The Representative Firm Theorem

When production sets are convex cones, firms do not really play much of a role in the economy—they earn zero profits in equilibrium, and it is actually irrelevant whether there is a single firm that possesses the entire set of activities $\mathcal{M}$ or a collection of M firms that each possess only a single activity a_m (plus the free disposal activities). The result that it is without loss of generality to focus on a single firm that possesses the sum of firms' production technologies is actually a much more general result than this example illustrates, as the following theorem highlights.

Theorem 12 (Representative Firm Theorem). *Let $\mathcal{E}$ be a production economy satisfying $(A5) - (A7)$. Given a price vector $p \in R^L_+$, denote the set of profit-maximizing net supplies of firm $j \in \mathcal{J}$ by $y_j(p)$. Then there exists a representative firm with production possibilities set $\mathcal{Y}$ and a set of profit-maximizing net supplies $y(p)$ such that $y^* \in y(p)$ if and only if $y^* = \sum_{j \in \mathcal{J}} y^*_j$ for some $y^*_j \in y_j(p)$ for each $j \in \mathcal{J}$.*

This theorem shows us that we can "aggregate" the production side of the economy. Exercise 14 asks you to prove this result. We know that when there are no income effects for consumers, we can represent all the consumers in the economy with a single representative consumer. The same idea holds for firms. But for firms, at least for the simplistic way we are currently thinking about firms, there are never income effects—prices have no impact on firms' *feasible* production plans. One implication of the representative firm theorem is that, if we take this simplistic view of firms, we are left with a somewhat simplistic view of the supply side of the economy as a whole. This point will be important for us to remember when we talk about the theory of the firm and why we might care about firm boundaries.

Exercise 14. This exercise asks you to prove the representative firm theorem.

(a) Fix p and construct $y^* = \sum_{j \in \mathcal{J}} y_j^*$ for some $y_j^* \in y_j(p)$ for each $j \in \mathcal{J}$. Prove that we must have $y^* \in y(p)$.

(b) Let $y^* \in y(p)$ be a profit-maximizing choice for the representative firm. Show that if $y^* = \sum_{j \in \mathcal{J}} y_j$ for some $y_j \in \mathcal{Y}_j$ for each $j \in \mathcal{J}$, then $y_j \in y_j(p)$ for each $j \in \mathcal{J}$.

10 Time and Uncertainty in General Equilibrium

Another, and perhaps the most important, extension to the general equilibrium framework is to allow for both time and uncertainty. Introducing time into the framework is straightforward: we can think of a consumption good today as a different commodity than a consumption good tomorrow. Adding uncertainty turns out also to be straightforward due to an important modeling device of Arrow (1953): states of the world. A state of the world is a complete description of a date-event. Everyone agrees on the set of possible states and what state of the world is realized, although they need not necessarily agree on the probabilities of those states occurring. This way of thinking about uncertainty makes it very easy to extend the general equilibrium framework. In fact, Debreu's (1959) *Theory of Value* devotes only a short chapter to general equilibrium under uncertainty, and in some sense the first paragraph of that chapter tells us the main idea.

> "The analysis is extended in this chapter to the case where uncertain events determine the consumption sets, the production sets, and the resources of the economy. A contract for the transfer of a commodity now specifies, in addition

to its physical properties, its location and its date, an event on the occurrence of which the transfer is conditional. This new definition of a commodity allows one to obtain a theory of uncertainty free from any probability concept and formally identical with the theory of certainty developed in the preceding chapters." (Debreu, 1959)

10.1 Arrow-Debreu Model

We will consider a parsimonious model of GE with uncertainty, although the framework can accommodate much more general specifications. Suppose there are I consumers $i \in \mathcal{I}$, L consumption goods $l \in \mathcal{L}$, and two periods, $t \in \{0, 1\}$. There are S possible states of the world that can occur at $t = 1$, $s \in \mathcal{S} = \{1, \ldots, S\}$. A consumption bundle for consumer i is an $x_i = (x_{0,i}, x_{1,i}, \ldots, x_{S,i})$, where $x_{0,i} = (x_{1,0,i}, \ldots, x_{L,0,i})$ is consumer i's consumption of the L goods at $t = 0$, and $x_{s,i} = (x_{1,s,i}, \ldots, x_{L,s,i})$ is her consumption of the L goods at $t = 1$ and in state s. Her consumption set is $\mathcal{X}_i = \mathbb{R}_+^{L(S+1)}$, and her preferences are given by her utility function $U_i : \mathbb{R}_+^{L(S+1)} \to \mathbb{R}$. She has endowment $\omega_i = (\omega_{0,i}, \omega_{1,i}, \ldots, \omega_{S,i})$, where $\omega_{0,i} = (\omega_{1,0,i}, \ldots, \omega_{L,0,i})$ is her endowment of the L goods at $t = 0$, and $\omega_{s,i} = (\omega_{1,s,i}, \ldots, \omega_{L,s,i})$ is her endowment of the L goods at $t = 1$ in state s.

Since there is uncertainty, we also have to specify consumers' beliefs. Suppose, at $t = 0$, consumer i believes that state $s \in \mathcal{S}$ will occur with probability $\pi_{s,i} \geq 0$, where $\sum_{s \in \mathcal{S}} \pi_{s,i} = 1$ for all i. Typically, we will think about consumers having the same beliefs, so that $\pi_{s,i} = \pi_s$ for all i, but the framework allows for subjective beliefs. We will also typically assume that consumers are expected utility maximizers with additively separable time preferences, so that we can write

$$U_i(x_i) = u_{0,i}(x_{0,i}) + \sum_{s \in \mathcal{S}} \pi_{s,i} u_{s,i}(x_{s,i}),$$

with $u_{0,i}$ and each of the $u_{s,i}$ functions concave.

In line with Debreu's description, we will think of each consumption good in each state of the world as being a separate commodity. To specify prices, therefore, we will have to specify $p = (p_0, p_1, \ldots, p_S)$, where $p_0 = (p_{1,0}, \ldots, p_{L,0}) \in \mathbb{R}^L$ is the price vector at $t = 0$, and $p_s = (p_{1,s}, \ldots, p_{L,s}) \in \mathbb{R}^L$ is the price vector at $t = 1$ in state $s \in \mathcal{S}$. That is, for a price $p_{l,s}$, consumers can buy and sell consumption of good l in state s.

There are three important assumptions that allow us to make use of all of the results we have derived so far in this course. The first assumption is that all trade occurs at time $t = 0$. So, at time $t = 0$, consumers buy and sell $t = 0$ commodities, and they also buy future claims to each commodity in each state of the world, and there is no opportunity for them to buy or sell at $t = 1$. The second important assumption is that the trading contracts that each consumer "writes" at $t = 0$ over $t = 1$ consumer are faithfully executed at $t = 1$. In the background, we are implicitly assuming the existence of an infallible third-party court system that perfectly compels consumers to execute their $t = 0$ contracts. This assumption, in turn, means that the third party can costlessly verify what state of the world was actually realized at $t = 1$. The third important assumption is that there is a market for each of the $L(S+1)$ state-contingent commodities.

Given prices p, consumer i solves

$$\max_{x_i \in \mathbb{R}_+^{L(S+1)}} U_i(x_i) \text{ s.t. } x_i \in \mathcal{B}_i(p),$$

where consumer i's budget set is given by

$$\mathcal{B}_i(p) = \left\{ x_i : p_0 \cdot x_{0,i} + \sum_{s \in \mathcal{S}} p_s \cdot x_{s,i} \leq p_0 \cdot \omega_{0,i} + \sum_{s \in \mathcal{S}} p_s \cdot \omega_{s,i} \right\}.$$

We will denote her Marshallian demand correspondence by $x_i(p, p \cdot x_i)$. A **pure-exchange economy with uncertainty** is therefore summarized by $\mathcal{E} = (u_i, \omega_i)_{i \in \mathcal{I}}$.

We are now in a position to define a Walrasian equilibrium in this context. For historical reasons, Walrasian equilibria in this model are referred to as Arrow-Debreu equilibria.

Definition 10. *An **Arrow-Debreu equilibrium** for pure-exchange economy with uncer-*

tainty $\mathcal{E}$ is a vector $\left(p^*, (x_i^*)_{i\in\mathcal{I}}\right)$ that satisfies:

1. *Consumer optimization: for all consumers $i \in \mathcal{I}$,*

$$x_i^* \in \underset{x_i \in \mathcal{B}_i(p^*)}{\operatorname{argmax}} U_i(x_i),$$

2. *Market-clearing: for all commodities $l \in \mathcal{L}$ and all $s \in \{0, 1, \ldots, S\}$,*

$$\sum_{i\in\mathcal{I}} x_{l,s,i}^* = \sum_{i\in\mathcal{I}} \omega_{l,s,i}.$$

This model is an elegant way of incorporating time and uncertainty into the basic framework because it allows us to apply all the results we have developed so far. For example, if $(A1) - (A4)$ hold, then a Walrasian equilibrium exists, and the welfare theorems hold.

The model does have some issues, though. One natural concern is that it seems unrealistic to think of all trading over future state-contingent commodities taking place at the beginning of time. Instead, we might expect that there would be different financial securities that are traded at potentially different times and these securities pay out when certain events occur. For example, car insurance pays out when your car is stolen, stocks pay dividends when a company is doing well, and so on.

Exercise 15. Consider a two date exchange economy with consumption at dates 0 and 1. There is a single consumer, one consumption good at each date, and there are S states of the world (realized at date 1).

The consumer's utility function is

$$U = u(x_0) + \delta \sum_{s=1}^{S} \pi_s u(x_s),$$

where x_0 is date 0 consumption, x_s is date 1 consumption in state s, u is "well-behaved," and $\delta \in (0,1)$. The consumer has initial endowment $(\omega_0, \omega_1, \ldots, \omega_S) \in \mathbb{R}_+^{S+1}$.

Write down the Arrow-Debreu equilibrium for this economy (normalize the price of date-0 consumption to be 1). Interpret the Arrow-Debreu relative prices: what factors determine whether they are high or low?

10.2 Sequential Trade and Arrow Equilibrium

Arrow later reformulated the model to allow for sequential trade in the following way. As before, there are two dates, $t \in \{0, 1\}$, and at date $t = 1$, a state of the world $s \in \mathcal{S}$ is realized. Suppose that consumption occurs only at $t = 1$, so that all consumers are endowed with $\omega_{l,0,i} = 0$ for all $l \in \mathcal{L}$, and $x_{0,i} \in \{0\}$ for all i. Moreover, suppose consumer i is an expected utility maximizer, so that $U_i(x_i) = \sum_{s \in \mathcal{S}} \pi_{s,i} u_{s,i}(x_{s,i})$.

At date $t = 0$, consumers cannot directly trade all $L \cdot S$ state-contingent commodities. They can, however, trade securities that pay off different amounts in different states at $t = 1$. In particular, they can trade S different Arrow securities, where at $t = 1$, **Arrow security** s pays \$1 if state s is realized, and it pays 0 otherwise. Each consumer is endowed with 0 of each Arrow security, but they can have positive or negative holdings of them after trade occurs at $t = 0$. Denote by $z_i = (z_{1,i}, \ldots, z_{S,i})$ consumer i's holdings of the S Arrow securities, and denote the price vector for the Arrow securities by $q = (q_1, \ldots, q_S)$. To anticipate how we will think of more general securities in the next section, denote the **dividends vector for security** k by $r_k = (r_{1,k}, \ldots, r_{S,k}) \in \mathbb{R}^S_+$, where $r_{s,k}$ is the amount that security k pays in state s. The dividends vector for Arrow security 1 is therefore $r_1^A \equiv (1, 0, \ldots, 0)$, and for Arrow security k is $r_k^A \equiv (0, \ldots, 0, 1, 0, \ldots, 0)$, where the kth element is 1 and all others are 0.

At $t = 1$, the state of the world s is realized, and then markets for each of the L goods open, consumers trade at prices $p_s = (p_{s,1}, \ldots, p_{s,L})$, and then they consume.

Under this specification, given Arrow security prices q and goods prices p, consumer i solves the following problem:

$$\max_{(z_i, x_i)} \sum_{s \in \mathcal{S}} \pi_{s,i} u_{s,i}(x_{s,i}) \text{ s.t. } (z_i, x_i) \in \mathcal{B}_i(q, p),$$

where her budget set is now given by

$$\mathcal{B}_i(q,p) = \{(z_i, x_i) : q \cdot z_i \leq 0, p_s \cdot x_{s,i} \leq p_s \cdot \omega_{s,i} + z_{s,i} \text{ for all } s \in \mathcal{S}\}.$$

The first inequality in the definition of the budget set reflects the assumption that the consumer is not endowed with any Arrow securities, so that the net value of the Arrow securities she holds after $t = 0$ trade has to be nonpositive. The second set of inequalities reflects her budget set at $t = 1$ in state s. Her wealth in state s is the sum of the wealth from her endowment, $p_s \cdot \omega_{s,i}$, and the wealth she obtains from her Arrow securities, $z_{s,i}$, which can be positive or negative. Note that, since we are assuming that $x_i \in \mathbb{R}_+^{L \cdot S}$, we are implicitly imposing the constraint that each consumer has nonnegative wealth at $t = 1$: $z_{s,i} \geq -p_s \cdot \omega_{s,i}$.

The idea behind this alternative setup of the model is that consumers will trade multiple times, and their wealth each time they trade is determined by their endowment in the "spot market" as well as how much they loaned and borrowed. Consumers correctly anticipate spot-market prices in each state at $t = 0$, even though they cannot trade in those markets until $t = 1$, and they buy and sell Arrow securities to transfer their wealth from one state to the next so they can buy the commodities they would like to buy in those states. We will refer to the economy as a **sequential-exchange economy with a complete set of Arrow securities** and denote it by $\mathcal{E}^{SE} = \left((u_i, \omega_i)_{i \in \mathcal{I}}, \left(r_k^A\right)_{k \in \mathcal{S}}\right)$, where r_k^A is the returns vector for the kth Arrow security.

We can now define our notion of Walrasian equilibrium in this setting.

Definition 11. *An* ***Arrow equilibrium*** *for a sequential-exchange economy with a complete set of Arrow securities,* $\mathcal{E}^{SE}$*, is a vector* $\left(q^*, p^*, (z_i^*, x_i^*)_{i \in \mathcal{I}}\right)$ *of Arrow security prices and state-contingent consumption-good prices and Arrow security positions and consumption bundles for each consumer* $i \in \mathcal{I}$ *that satisfies*:

1. *Consumer optimization: for all consumers* $i \in \mathcal{I}$,

$$(z_i^*, x_i^*) \in \underset{(z_i,x_i)\in\mathcal{B}_i(q,p)}{\text{argmax}} \sum_{s\in\mathcal{S}} \pi_{s,i} u_{s,i}\left(x_{s,i}\right),$$

2. *Market-clearing:* $\sum_{i\in\mathcal{I}} z_i^* = 0$ *and, for all commodities* $l \in \mathcal{L}$ *and all* $s \in \mathcal{S}$,

$$\sum_{i\in\mathcal{I}} x_{l,s,i}^* = \sum_{i\in\mathcal{I}} \omega_{l,s,i}.$$

Given this definition of equilibrium, we can now describe the main result of this section, which links the set of allocations that can arise in an Arrow-Debreu equilibrium in a pure-exchange economy with uncertainty, $\mathcal{E} = (u_i, \omega_i)_{i\in\mathcal{I}}$ to the set of allocations that can arise in an Arrow equilibrium in a sequential-exchange economy with a complete set of Arrow securities, $\mathcal{E}^{SE} = \left((u_i, \omega_i)_{i\in\mathcal{I}}, \left(r_k^A\right)_{k\in\mathcal{S}}\right)$.

Theorem 13 (Equivalence of Arrow and Arrow-Debreu equilibrium). *Given economies* $\mathcal{E}$ *and* $\mathcal{E}^{SE}$ *with the same consumer preferences and endowments,* $(x_i^*)_{i\in\mathcal{I}}$ *is an Arrow-Debreu equilibrium allocation for* $\mathcal{E}$ *if and only if, for some* $(z_i^*)_{i\in\mathcal{I}}$, $(z_i^*, x_i^*)_{i\in\mathcal{I}}$ *is an Arrow equilibrium allocation for* $\mathcal{E}^{SE}$.

Proof of Theorem 13. Take an Arrow equilibrium $\left(q^*, p^*, (z_i^*, x_i^*)_{i\in\mathcal{I}}\right)$ for economy $\mathcal{E}^{SE}$. By monotonicity of preferences, we will have that for each consumer $i \in \mathcal{I}$, $q^* \cdot z_i^*\left(q^*, p^*\right) = 0$ and $z_{s,i}^*\left(q^*, p^*\right) = p_s^* \cdot x_{s,i}^*\left(q^*, p^*\right) - p_s^* \cdot \omega_{s,i}$. We can combine these two equations to get

$$\sum_{s\in\mathcal{S}} q_s^* \left(p_s^* \cdot x_{s,i}^* - p_s^* \cdot \omega_{s,i}\right) = 0.$$

The consumption bundle $x_{s,i}^*$ therefore solves the problem:

$$\max_{x_i} \sum_{s\in\mathcal{S}} \pi_{s,i} u_{s,i}\left(x_{s,i}\right)$$

subject to

$$\sum_{s\in\mathcal{S}}(q_s^* p_s^*)\cdot x_{s,i}\leq\sum_{s\in\mathcal{S}}(q_s^* p_s^*)\cdot\omega_{s,i}.$$

Define state-dependent prices $\hat{p}_s^* \in \mathbb{R}^L$ for $s \in \mathcal{S}$ with $\hat{p}_s^* \equiv q_s^* p_s^*$. Then, $\left(\hat{p}^*, (x_i^*)_{i\in\mathcal{I}}\right)$ is an Arrow-Debreu equilibrium for economy $\mathcal{E}$.

Going the other direction, suppose $\left(p^*, (x_i^*)_{i\in\mathcal{I}}\right)$ is an Arrow-Debreu equilibrium for $\mathcal{E}$. Then $\left(q^*, \hat{p}^*, (z_i^*, x_i^*)_{i\in\mathcal{I}}\right)$, where $q_s^* = 1$, $\hat{p}^* = p^*$, and

$$z_{s,i}^* = p_s^* \cdot x_{s,i}^* - p_s^* \cdot \omega_{s,i}$$

is an Arrow equilibrium for economy $\mathcal{E}^{SE}$.■

This theorem establishes an equivalence between the notion of Arrow-Debreu equilibrium in which trade in all $L \cdot S$ markets occurs ex ante and the notion of Arrow equilibrium, in which trade occurs in only S markets at $t = 0$ and in L markets at $t = 1$. One disadvantage of the notion of Arrow equilibrium is that even though trading seems less complicated, in a sense, consumers still must form consistent expectations about what the equilibrium goods prices will be at $t = 1$ when they are trading securities at $t = 0$. That said, one of the big advantages of the sequential exchange framework is that it allows us to investigate what happens when the $t = 0$ securities market does not have a complete set of Arrow securities. That is, what happens if markets are *incomplete*? We will turn to this question now.

Exercise 16. There are two farmers, named Octavia and Seema, who can trade only with each other. In years when there is no flood, both farms yield 10 units of corn; in years when there is a flood, Octavia's farm yields 10 units of corn, and Seema's farm yields 5 units of corn. The probability of a flood is given by $\pi = 1/2$, which is common knowledge to the farmers. The farmers have identical utility functions given by $u(x) = \ln(x)$, where x is the units of corn consumed.

(a) Suppose that Octavia and Seema set up an exchange market to securitize corn at the beginning of the year (before knowing the realization of the state of the world). Compute the equilibrium prices and allocations.

Suppose that Seema has the option of building a greenhouse at a cost before realizing the state of the world. If she builds a greenhouse, Seema's farm will produce 10 units of corn in all states of the world.

(b) Using the equilibrium results computed above, how much would each farmer be willing to pay for the greenhouse? Assume that each considers paying for the greenhouse entirely by herself. In this context, should we consider the possibility of "negative" willingness-to-pay? That is, might one farmer be willing to pay the other not to build the greenhouse?

(c) Would your above answer change if Octavia and Seema were unable to trade ex post (after the state of the world is realized)? If so, how? Would your answer change if they were unable to trade ex ante (there is no exchange market to securitize corn at the beginning of the year)?

10.3 Incomplete Markets

When we talked about sequential exchange economies in the previous section, we assumed that there was a complete set of Arrow securities that could be traded at $t = 0$. One important implication of this assumption that we did not emphasize is that it allowed consumers to insure themselves against the state of the world by transferring wealth from states in which their marginal utility of income is low (either because they do not especially value consumption in such states or because their endowment in such states is high) to states in which their marginal utility of income is high. In an Arrow equilibrium, the resulting risk-sharing is efficient, since the first welfare theorem applies in that setting.[7] In contrast, when markets are *incomplete*, risk sharing in the economy will generally be inefficient. This will imply that Walrasian equilibrium allocations in such economies are not Pareto optimal.

Suppose there are K **securities** $k \in \mathcal{K} = \{1, \ldots, K\}$, where security k has **dividends vector** $r_k = (r_{1,k}, \ldots, r_{S,k}) \in \mathbb{R}^S_+$. We can think of each security as a share in a company that pays out dividends $r_{s,k}$ in state s. If consumer i owns **portfolio** $z_i = (z_{1,i}, \ldots, z_{K,i})$, then in state s, her wealth will be $p_s \cdot \omega_{s,i} + \sum_{k \in \mathcal{K}} z_{k,i} r_{s,k}$. If we denote the **dividends matrix** $R = \left(r_1^T, \ldots, r_K^T\right)$, where r_k^T is the transpose of r_k, then we will say that the securities market is **incomplete** if $rank\,(R) < S$. Otherwise, we will say that the securities market

[7]More precisely, define $v_{s,i}\,(z_{s,i}) = \max_{x_{s,i}} u_{s,i}\,(x_{s,i})$ subject to $p_s \cdot x_{s,i} \leq p_s \cdot \omega_{s,i} + z_{s,i}$ to be consumer i's indirect utility in state s when she has $z_{s,i}$ units of Arrow security s. If we assume $v_{s,i}$ is concave and differentiable, then Pareto optimality ensures that for all i, i', $\frac{\pi_{s,i} \partial v_{s,i} / \partial z_{s,i}}{\pi_{s,i'} \partial v_{s,i'} / \partial z_{s,i'}} = \frac{\lambda_i}{\lambda_{i'}}$ for all s. That is, the ratio of marginal utilities of income are equalized across states for any two consumers. By the first welfare theorem, any Arrow equilibrium allocation satisfies these properties.

is **complete**. If there is a complete set of Arrow securities, then R is the $S \times S$ identity matrix, and the securities market is complete.

As in the previous subsection, given security prices $q = (q_1, \ldots, q_K)$ and goods prices p, consumer $i \in \mathcal{I}$ solves the following problem:

$$\max_{(z_i, x_i)} \sum_{s \in \mathcal{S}} \pi_{s,i} u_{s,i}(x_{s,i}) \text{ s.t. } (z_i, x_i) \in \mathcal{B}_i(q, p),$$

where her budget set is now given by

$$\mathcal{B}_i(q, p) = \left\{ (z_i, x_i) : q \cdot z_i \leq 0, p_s \cdot x_{s,i} \leq p_s \cdot \omega_{s,i} + \sum_{k \in \mathcal{K}} z_{k,i} r_{s,k} \text{ for all } s \in \mathcal{S} \right\}.$$

As before, consumers maximize their expected utility subject to a $t = 0$ budget constraint and a $t = 1$ budget constraint for each state $s \in \mathcal{S}$. The first inequality in the definition of the budget set again reflects the assumption that the consumer is not endowed with any securities.

A **sequential-exchange economy with securities** $\mathcal{K}$ is summarized by a vector $\mathcal{E}^{SE} = \left((u_i, \omega_i)_{i \in \mathcal{I}}, R \right)$. We can now define our notion of Walrasian equilibrium for such an economy.

Definition 12. *An **incomplete-markets equilibrium** (or **Radner equilibrium**) for a sequential-exchange economy with securities $\mathcal{K}$ is a vector $\left(q^*, p^*, (z_i^*, x_i^*)_{i \in \mathcal{I}}\right)$ of security prices and state-contingent consumption-good prices and security positions and consumption bundles for each consumer $i \in \mathcal{I}$ that satisfies*:

1. *Consumer optimization: for all consumers $i \in \mathcal{I}$,*

$$(z_i^*, x_i^*) \in \underset{(z_i, x_i) \in \mathcal{B}_i(q,p)}{\operatorname{argmax}} \sum_{s \in \mathcal{S}} \pi_{s,i} u_{s,i}(x_{s,i}),$$

2. *Market-clearing: $\sum_{i \in \mathcal{I}} z_{k,i}^* = 0$ for all $k \in \mathcal{K}$, and, for all commodities $l \in \mathcal{L}$ and all*

$s \in \mathcal{S}$,

$$\sum_{i \in \mathcal{I}} x^*_{l,s,i} = \sum_{i \in \mathcal{I}} \omega_{l,s,i}.$$

In general, when markets are incomplete, a Radner equilibrium need not exist, and even if it does exist, the resulting allocation will typically not be Pareto optimal. If, however, $L = 1$, so there is only a single consumption good, then a Radner equilibrium exists, and it does have some optimality properties.

For $L = 1$, Diamond (1967) showed that a Radner equilibrium exists by showing that the consumer optimization problem boils down to a more familiar problem. In particular, at any solution to consumer i's problem, we must have $q \cdot z_i = 0$ and $x_{s,i} = \omega_{s,i} + \sum_{k \in \mathcal{K}} z_{k,i} r_{s,k}$. We can substitute this second constraint into the consumer's problem, which then becomes

$$\max_{z_i} \underbrace{\sum_{s \in \mathcal{S}} \pi_{s,i} u_{s,i} \left(\omega_{s,i} + \sum_{k \in \mathcal{K}} z_{k,i} r_{s,k} \right)}_{\equiv \tilde{u}_i(z_i)}$$

subject to $q \cdot z_i \leq 0$. Diamond pointed out that such an economy is equivalent, in some sense, to an economy in which consumer preferences are given by $\tilde{u}_i(z_i)$ and there are K "commodities"—one corresponding to each of the securities. So as long as $\tilde{u}_i$ satisfies $(A1)$ – $(A3)$, then a WE exists. The interior endowments assumption $(A4)$ is not necessary for the existence result because consumers are allowed to "consume" negative quantities of z_i.

Such equilibria need not yield Pareto-optimal allocations. To see why, consider an example in which $L = 1, K = 1$, and $S = 2$. The security pays 1 in each state of the world. There are two consumers with endowments $\omega_1 = (2, 1)$ and $\omega_2 = (1, 2)$, so that consumer 1 is endowed with one unit of the consumption good in state 1 and 2 units in state 2. Both consumers have identical preferences given by

$$u_i(x_{1,i}, x_{2,i}) = \frac{1}{2} \log x_{1,i} + \frac{1}{2} \log x_{2,i}.$$

As an exercise, it is worth verifying that there is a unique Radner equilibrium of this economy. In this equilibrium, there will be no trade in the security at $t = 0$, and consumers will consume their endowments at $t = 1$. This allocation is Pareto dominated by the feasible allocation $x_1 = x_2 = (3/2, 3/2)$.

The first welfare theorem fails in this situation because the set of existing securities does not allow the consumers to insure themselves against states in which they will have a low endowment. Nevertheless, there is still a sense in which the resulting allocation exhausts the gains from trade and is therefore what we refer to as constrained efficient.

Definition 13. *Given endowments $(\omega_i)_{i\in\mathcal{I}}$ and securities $\mathcal{K}$, an allocation $(x_i)_{i\in\mathcal{I}}$ is **constrained efficient** if $\sum_{i\in\mathcal{I}} x_i \leq \sum_{i\in\mathcal{I}} \omega_i$, and for all i, there exists $z_i \in R^K$ such that $x_i = \omega_i + Rz_i$, and there exists no alternative allocation $(\hat{x}_i)_{i\in\mathcal{I}}$ that Pareto dominates $(x_i)_{i\in\mathcal{I}}$ and also satisfies $\sum_{i\in\mathcal{I}} \hat{x}_i \leq \sum_{i\in\mathcal{I}} \omega_i$ and $\hat{x}_i = \omega_i + Rz_i$ for some $z_i \in \mathbb{R}^K$ for all i.*

When there is only a single consumption good and consumers have monotone preferences, Radner equilibrium allocations are always constrained efficient, as the following theorem illustrates.

Theorem 14. *If $\mathcal{E}^{SE}$ has $L = 1$ and satisfies (A2), then if $\left(q^*, p^*, (z_i^*, x_i^*)_{i\in\mathcal{I}}\right)$ is a Radner equilibrium, $(x_i^*)_{i\in\mathcal{I}}$ is constrained efficient.*

We will conclude this section with a few comments on the generality of this theorem. As Hart (1975) illustrates, when $L = 2$, there may exist Radner equilibria that are not constrained efficient (see, for instance, MWG Example 19.F.2). Also, when markets are incomplete, weird things can happen. For example, adding another security that is linearly independent of existing securities can actually make all consumers strictly worse off (see, for instance, MWG Exercise 19.F.3). Finally, when markets are incomplete, Geanakoplos and Polemarchakis (1986) shows that it is generically true that a social planner can improve efficiency by introducing a small tax or subsidy. This is an illustration of the "general theory of the second-best" (Lipsey and Lancaster, 1963): when there is an unresolvable market failure (market incompleteness, in this case), it is generically the case that there exists a further distortion that a social planner could conceivably put in place that leads to a more efficient allocation.