Economics 2010b Problem Set 3

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Due: Friday, 2/16 by 11:59pm

Exercise 12. Consider an economy with two consumers and two commodities. Consumer 1's endowment vector is $(\lambda, 0)$ and consumer 2's is $(\mu, 0)$. Each consumer's utility is the sum of their consumption of the two commodities. Consumer 1 owns a technology for transforming commodity 1 into commodity 2. The production function is $Y = X^2$, where X is the input of commodity 1.

(a) Does this economy have a Walrasian equilibrium?

(b) What allocation would a planner choose to maximize the sum of utilities? [Be careful about second-order conditions.]

(c) What is the core of this economy?

Exercise 13 (Adapted from MWG, 16.F.2-4). In the first week, we discussed the first-order conditions for Pareto optimality in exchange economies. This exercise asks you to extend these conditions to production economies with I consumers and J firms. Define the utility possibility set:

$$\mathcal{U} = \left\{(u_1, \ldots, u_I) \in \mathbb{R}^I : \exists \text{ feasible } (x_i)_{i\in\mathcal{I}}, (y_j)_{j\in\mathcal{J}} \text{ with } u_i(x_i) \geq u_i \text{ for all } i\right\}.$$

Assume the production set for firm j takes the form $\mathcal{Y}_j = \left\{y \in \mathbb{R}^L : F_j(y) \leq 0\right\}$, where $F_j(y) = 0$ defines firm j's **transformation frontier**, and $F_j : \mathbb{R}^L \to \mathbb{R}$ is twice continuously differentiable with $F_j(0) \leq 0$ and $\nabla F_j(y_j) >> 0$ for all $y_j \in \mathbb{R}^L$.

(a) Show that if F_j is a convex function, then $\mathcal{Y}_j$ is a convex set.

(b) [Optional—simultaneously fun and tedious] Show that if, for all $i \in \mathcal{I}$, $\mathcal{X}_i$ is convex and u_i is concave, and for all $j \in \mathcal{J}$, F_j is convex, then $\mathcal{U}$ is a convex set.

(c) Suppose $\mathcal{U}$ is a convex set. Suppose $\lambda \geq 0$ is a non-zero vector of Pareto weights, and consider the Pareto problem

$$\max_{u\in\mathcal{U}} \lambda \cdot u.$$

Show that the optimality conditions for an interior solution (i.e. $x_i >> 0$ for all i) for this problem satisfy

$$\frac{\partial u_i/\partial x_{l,i}}{\partial u_i/\partial x_{l',i}} = \frac{\partial u_{i'}/\partial x_{l,i'}}{\partial u_{i'}/\partial x_{l',i'}} \text{ for all } i, i', l, l' \quad (1)$$

$$\frac{\partial F_j/\partial y_{l,j}}{\partial F_j/\partial y_{l',j}} = \frac{\partial F_{j'}/\partial y_{l,j'}}{\partial F_{j'}/\partial y_{l',j'}} \text{ for all } j, j', l, l' \quad (2)$$

$$\frac{\partial u_i/\partial x_{l,i}}{\partial u_i/\partial x_{l',i}} = \frac{\partial F_j/\partial y_{l,j}}{\partial F_j/\partial y_{l',j}} \text{ for all } i, j, l, l'. \quad (3)$$

(d) Consider the aggregate problem of maximizing the production of commodity 1 subject to minimum production levels $(\bar{y}_2, \ldots, \bar{y}_L)$ for the other commodities.

$$\max_{(y_1,\ldots,y_J)} \sum_{j\in\mathcal{J}} y_{1,j}$$

subject to

$$\sum_{j\in\mathcal{J}} y_{l,j} \geq \bar{y}_l \text{ for all } l = 2, \ldots, L$$

and

$$F_j(y_j) \leq 0 \text{ for all } j = 1, \ldots, J.$$

Show that the optimality conditions for this problem satisfy (2). What does these conditions imply about how production is carried out across firms in a Pareto optimal allocation?

Exercise 14. This exercise asks you to prove the representative firm theorem. Recall the representative firm theorem:

Theorem (Representative Firm Theorem). Let $\mathcal{E}$ be a production economy satisfying $(A5) - (A7)$. Given a price vector $p \in \mathbb{R}^L_+$, denote the set of profit-maximizing net supplies of firm $j \in \mathcal{J}$ by $y_j(p)$. Then there exists a representative firm with production possibilities set $\mathcal{Y}$ and a set of profit-maximizing net supplies $y(p)$ such that $y^* \in y(p)$ if and only if $y^* = \sum_{j\in\mathcal{J}} y_j^*$ for some $y_j^* \in y_j(p)$ for each $j \in \mathcal{J}$.

(a) Fix p and construct $y^* = \sum_{j\in\mathcal{J}} y_j^*$ for some $y_j^* \in y_j(p)$ for each $j \in \mathcal{J}$. Prove that we must have $y^* \in y(p)$.

(b) Let $y^* \in y(p)$ be a profit-maximizing choice for the representative firm. Show that if $y^* = \sum_{j\in\mathcal{J}} y_j$ for some $y_j \in \mathcal{Y}_j$ for each $j \in \mathcal{J}$, then $y_j \in y_j(p)$ for each $j \in \mathcal{J}$.

Exercise 15. Consider a two-date exchange economy with consumption at dates 0 and 1. There is a single consumer, one consumption good at each date, and there are S states of the world (realized at date 1).

The consumer's utility function is

$$U = u(x_0) + \delta \sum_{s=1}^{S} \pi_s u(x_s),$$

where x_0 is date 0 consumption, x_s is date 1 consumption in state s, u is "well-behaved," and $\delta \in (0,1)$. The consumer has initial endowment $(\omega_0, \omega_1, \ldots, \omega_S) \in \mathbb{R}^{S+1}_+$.

Write down the Arrow-Debreu equilibrium for this economy (normalize the price of date-0 consumption to be 1). Interpret the Arrow-Debreu relative prices: what factors determine whether they are high or low?

Exercise 16. There are two farmers, named Octavia and Seema, who can trade only with each other. In years when there is no flood, both farms yield 10 units of corn; in years when there is a flood, Octavia's farm yields 10 units of corn, and Seema's farm yields 5 units of corn. The probability of a flood is given by $\pi = 1/2$, which is common knowledge to the farmers. The farmers have identical utility functions given by $u(x) = \ln(x)$, where x is the units of corn consumed.

(a) Suppose that Octavia and Seema set up an exchange market to securitize corn at the beginning of the year (before knowing the realization of the state of the world). Compute the equilibrium prices and allocations.

Suppose that Seema has the option of building a greenhouse at a cost before realizing the state of the world. If she builds a greenhouse, Seema's farm will produce 10 units of corn in all states of the world.

(b) Using the equilibrium results computed above, how much would each farmer be willing to pay for the greenhouse? Assume that each considers paying for the greenhouse entirely by herself. In this context, should we consider the possibility of "negative" willingness-to-pay? That is, might one farmer be willing to pay the other not to build the greenhouse?

(c) Would your above answer change if Octavia and Seema were unable to trade ex post (after the state of the world is realized)? If so, how? Would your answer change if they were unable to trade ex ante (there is no exchange market to securitize corn at the beginning of the year)?